DIRECTED AND NON-DIRECTED PATH CONSTRAINED LAST-PASSAGE PERCOLATION

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Abstract. Hammersley's Last Passage Percolation (LPP), also known as Ulam's problem, is a well-studied model that can be described as follows: consider m points chosen uniformly and independently in $[0,1]^2$, then what is the maximal number $\mathcal{L}_m$ of points that can be collected by an up-right path? We introduce here a generalization of this standard LPP, in order to allow for more general constraints than the up-right condition (a 1-Lipschitz condition after rotation by 45°). We focus more specifically on two cases: (i) when the constraint comes from the γ-Hölder norm of the path (a local condition), we call it H-LPP; (ii) when the constraint comes from the entropy of a path (a global condition), we call it E-LPP. These generalizations of the standard LPP also allows us to deal with *non-directed* LPP. We develop motivations for directed and non-directed path-constrained LPP, and we find the correct order of $\mathcal{L}_m$ in a general manner – as a specific example, the maximal number of points that can be collected by a non-directed path of total length smaller than 1 is shown to be of order $\sqrt{m}$. This new LPP opens the way for many interesting problems, and we present some of its potential applications, to the context of directed and non-directed polymers in random environment. Several problems remain open.

Keywords: Last-passage percolation, non-directed polymers.

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1. Introduction

In this introduction, we recall the original Hammersley's LPP of the maximal number of points that can be collected by up/right paths, also known as Ulam's problem [25] of the maximal increasing subsequence of a random permutation. This problem has been the object of an intense activity over the past decades, culminating with the proof that it is exactly solvable, and in the so-called KPZ universality class. We show how to generalize this process by enlarging the set of paths allowed to collect points, by changing the *increasing* constraint (or a 1-Lipschitz constraint, by a 45° rotation), to a more general *compatibility* condition. We point out that the compatibility condition in the Hammersley's LPP is *local*, that is, the constraint to collect points depends only on two consecutive points. Conversely, a *global* condition is a constraint that takes in account the whole path trajectory that collects points.

In Section 2, we introduce some specific constraints of interest (local and global) in the directed setting and we derive the correct order for the LPP problems. In Section 3, we define a natural framework to be able to consider non-directed LPP and we also derive its correct order. Let us stress that in this article we put forward the interest of these models,

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focusing on simpler results in order to provide motivations for our study. The techniques we use are robust, and our results already have many possible applications, as seen in [6] or as developed in Section 4, where as one example we relate our results to the Hamilton-Jacobi equation considered in [3] to study stationary solutions for the Burgers equation. For this reason we do not pursue for optimal constants or for more precise convergence results, since it would bring many technicalities, and since it would dilute our core message. We conclude the paper by presenting some simulations, which help us to formulate a few conjectures on the convergence of the models, see Appendix A.

1.1. **Hammersley's Last Passage Percolation.** Let us take m points independently as uniform random variables in the square $[0,1]^2$, and denote the coordinates of these points $Z_1 := (t_1, x_1), Z_2 := (t_2, x_2)$, etc... We say that a sequence $(z_{i_\ell})_{1\leqslant \ell \leqslant k}$ is *increasing* if $t_{i_\ell} > t_{i_{\ell-1}}$ and $x_{i_\ell} > x_{i_{\ell-1}}$ for any $1 \leqslant \ell \leqslant k$ (we set by convention $i_0 = 0$ and $z_0 = (0,0)$).

Then, the question is to study the length of the longest increasing sequence among the m points which is equivalent to the length of the longest increasing subsequence of a random (uniform) permutation of length m. We denote:

$$\mathcal{L}_m := \sup\big\{k\ ; \exists\, (i_1,\dots,i_k)\ s.t.\ (Z_{i_\ell})_{1\leqslant \ell\leqslant k} \text{ is increasing}\big\}$$

Using subadditive techniques, Hammersley [13] first proved that $m^{-1/2}\mathcal{L}_m$ converges a.s. and in L^1 to some constant, that was believed to be 2. Further works then proven that the constant was indeed 2 [19, 26]. Moreover, and quite remarkably, this model has been shown to be exactly solvable by Baik, Deift and Johansson [2], and they identified the fluctuations of $\mathcal{L}_m$ around $2\sqrt{m}$, showing that the model is in the so-called KPZ universality class. More precisely, in [2] the authors showed the following result.

Theorem 1.1 ([2])**.** *We have the convergence in distribution*

$$\frac{\mathcal{L}_m - 2\sqrt{m}}{m^{1/6}} \xrightarrow{(d)} F_{GUE}\,,$$

where F_{GUE} is the Tracy-Widom GUE distribution.

Moreover, Johansson [17] proved that the typical transversal fluctuations of a path collecting the maximal number of points is of order $m^{-1/6}$.

Remark 1.2. Let us stress that the context of [17] is actually slightly different: Johansson considers up-right paths going from $(0,0)$ to (N,N) in a Poisson Point process of intensity 1: he shows that the typical transversal fluctuations (away from the diagonal) of a path collecting the maximal number of points is of order $N^{2/3}$. One recovers the setting presented above after rescaling by $1/N$ to reduce to $[0,1]^2$, with a Poisson point process of intensity $m = N^2$ instead of a fixed number m of points: it therefore tells that the transversal fluctuations of a maximal path is of order $N^{-1}N^{2/3} = m^{-1/6}$.

Let us also mention that in [11], the case when the points are not chosen uniformly in $[0,1]^2$ but have some given density $p(x,y)$ has also been solved: the limiting constant $\lim_{n\to\infty} \mathcal{L}_m/\sqrt{m}$ and the limiting curve are identified.

1.2. **General definition of path-constrained Last Passage Percolation.** We now perform a 45 degree clockwise rotation, and generalize Hammersley's LPP by introducing a general constraint on paths (that can be either *local* or *global*): we introduce it via a notion of *compatibility* of the points that can be collected. We need three ingredients:

- a domain Λ;

- a (finite or countable) random set of points $\Upsilon \subset \Lambda$, whose elements are denoted by $Z_i = (t_i, x_i)$ and its law is denoted $\mathbb{P}$;
- a compatibility condition, *i.e.* a set $\mathcal{C}$ of compatible subsets of Λ.

Then, we define the *$\mathcal{C}$-compatible Last-Passage Percolation* as the maximal number of $\mathcal{C}$-compatible points in Υ, that is

$$\mathcal{L}_{\Upsilon}^{(\mathcal{C})} = \mathcal{L}_{\Upsilon}^{(\mathcal{C})}(\Lambda) := \sup\Big\{|\Delta|\, ;\, \Delta \subset \Upsilon, \Delta \in \mathcal{C}\Big\}. \tag{1.1}$$

Remark 1.3. This fits the definition of Hammersley's LPP as defined above: the compatibility set $\mathcal{C}$ being the set of all increasing subsets of $[0,1]^2$. We can also define it, in an equivalent manner, after a rotation by 45°: we take the domain $\Lambda := \big\{(x,y), 0 < x < \sqrt{2}, |y| \leqslant \min(1, 1-t)\big\}$, and we use $\Upsilon = \Upsilon_m$ a set of m independent uniform random variables in Λ. The compatibility set is then taken to be (with the convention $(t_0, x_0) = (0,0)$)

$$\mathcal{C} = \bigcup_{k\geqslant 0} \Big\{\Delta = \{(t_i,x_i)\}_{1\leqslant i\leqslant k}\, ; 0 < t_1 < \cdots < t_k < \sqrt{2},\ \frac{|x_i - x_{i-1}|}{|t_i - t_{i-1}|} \leqslant 1 \text{ for all } 1 \leqslant i \leqslant k\Big\},$$

which corresponds to sets of points that can be collected via a 1-Lipschitz function. The Poissonian (point-to-point) version of Hammersley's also LPP can also be recovered by considering Υ a Poisson point process on $\mathbb{R}^2$ with intensity $\lambda > 0$, and $\Lambda = [0,t] \times \mathbb{R}$, with the same 1-Lipschitz compatibility condition as above.

Now, there are at least two reasonable ways of defining the compatibility condition: (i) by replacing the Lipschitz condition by a Hölder constraint; (ii) by considering an entropy constraint (a global constraint on the path, for instance on its Sobolev norm), that also allows to deal with non-directed paths. We restrict ourselves to the case of the dimension $d = 2$ for the simplicity of the exposition, but all our definitions and reasonings can easily be extended to the case of higher dimensions. We start with the case of directed paths in Section 2, and then discuss the non-directed case in Section 3. We present some potential applications in Section 4.

Several other constraints can be (and have been) considered, and let us mention a few. For instance the constraint that the path is convex has been studied in [1], and is related to the question of counting the number of lattice convex shapes, see [5, 24, 27] and more recently in [7]. The question of pattern-avoiding permutation has also gained some interest recently, see in particular [14, 20, 21]. Thinking about a polymer model, one may also think of a local "flexibility" condition for the set $\Delta = (\Delta_i)_{1\leqslant i\leqslant k}$ by considering the constraint $0 \leqslant \theta^{(1)} \leqslant \inf_i |\theta_i| \leqslant \sup_i |\theta_i| \leqslant \theta^{(2)} < \infty$, with θ_i the angle between the segments $[\Delta_{i-1}, \Delta_i]$ and $[\Delta_i, \Delta_{i+1}]$ (and $\theta_0 := 0$). This would model the stiffness of the polymer. In this paper we do not pursue in this direction.

We also mention that in [28], the author considers a related problem: the question is to obtain criteria for the existence of "regular functions" $f : \mathbb{R} \to \mathbb{R}$ (with several type of constraints, such as continuity, bounded variations, etc...) whose graph interpolates between random subsets of parallel vertical lines. This can be thought as a first passage percolation analogue to our problem, with a different distribution for the set of points considered.

2. Directed LPP: Hölder and entropy constraints

In this section, we consider directed paths. We work with a domain $\Lambda_{t,x} = [0,t] \times [-x,x]$, for some (fixed) $t, x > 0$. Then, we consider m independent r.v. uniform in $\Lambda_{t,x}$ to form the set Υ_m. We will use $\mathcal{L}_m$ as a short notation for $\mathcal{L}_{\Upsilon_m}$. Moreover, we say that a set

$\Delta = \{(t_i, x_i)\}_{1 \leqslant i \leqslant k} \subset \mathbb{R}_+ \times \mathbb{R}$ is *directed* if $0 < t_1 < \cdots < t_k$. We deal first with the local constraint (Hölder constraint), before we turn to the global one (Entropy constraint).

2.1. **Local Hölder constraint.** The first natural generalization of the 1-Lipschitz condition is to consider a Hölder constraint instead – the constraint is *local*, it depends only on two consecutive points. For any $\gamma \geqslant 0$, we can define the γ-Hölder norm of a set $\Delta = (t_i, x_i)_{1 \leqslant i \leqslant k}$ (in which the points are ordered $t_1 < \cdots < t_k$, with the convention $(t_0, x_0) = (0, 0)$)

$$\mathrm{H}_\gamma(\Delta) := \sup_{1 \leqslant i \leqslant k} \frac{|x_i - x_{i-1}|}{|t_i - t_{i-1}|^\gamma}. \tag{2.1}$$

Notice that this is not the γ-Hölder norm of the linear interpolation of the points, since (2.1) only considers consecutive points: one can think of this quantity as a *local* γ-Hölder norm. In particular, the case $\gamma > 1$ is not trivial here, and the case $\gamma = 0$ is also of interest. Then, for some fixed $A > 0$, we define a compatibility set

$$\mathcal{H}_\gamma^A := \left\{ \Delta \subset \mathbb{R}_+ \times \mathbb{R} \, ; \, \Delta \text{ directed}, \, \mathrm{H}_\gamma(\Delta) \leqslant A \right\}. \tag{2.2}$$

We then consider the γ-Hölder Last Passage Percolation, abbreviated as H_γ-LPP, defined as

$$\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x}) := \sup \left\{ |\Delta| \, ; \, \Delta \subset \Upsilon_m, \Delta \in \mathcal{H}_\gamma^A \right\}. \tag{2.3}$$

We prove the following result.

Theorem 2.1. *There are constants c_1, c_2 (depending only on γ, during the course of the proof one finds that $c_1 \leqslant c(1+\gamma)^{-1/2}$) such that for any t, x and B, for any $1 \leqslant k \leqslant m$*

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x}) \geqslant k \Big) \leqslant \Big(\frac{c_1 A t^\gamma m}{x k^{1+\gamma}} \Big)^k, \tag{2.4}$$

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x}) \leqslant k \Big) \leqslant \exp \Big\{ c_2 k \Big(1 - c_2 \Big(\frac{A t^\gamma}{x k^\gamma} \wedge 1 \Big) \frac{m}{k} \Big) \Big\}. \tag{2.5}$$

As a consequence, there is some $C > 0$ such that for any fixed t, x, γ, A, $\mathbb{P}$-a.s. there is some m_0 such that

$$\frac{1}{C} \leqslant \frac{\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x})}{(A t^\gamma / x)^{1/(1+\gamma)} m^{1/(1+\gamma)}} \leqslant C \qquad \textit{for all } m \geqslant m_0 .$$

We stress that the constants in (2.4)-(2.5) are uniform in the parameters m, A, t, x: the results are still valid when considering the situation when $A, t, x \to \infty$ as $m \to \infty$, which is useful for some applications. Note that we could define a point-to-point version of the H_γ-LPP, by adding the condition that $(t, 0) \in \Delta$: a result analogous to Theorem 2.1 then holds.

Note that we have that $\mathcal{L}_m^{(\mathcal{H}_\gamma^A)} = \mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x})$ is of order m^κ, with $\kappa = 1/(1+\gamma)$. Then, it is very natural to expect that $\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}/m^\kappa$ converges a.s. to a constant as $m \to \infty$: we discuss this convergence in Section 2.3, see in particular Remark 2.7. The value of the constant is discussed in Appendix A.

Let us also discuss briefly about the (conjectured) transversal fluctuations of a maximal path (that is a path collecting the maximal number of points). We already have that $\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}$ is of order m^κ, with κ going to 1 as $\gamma \downarrow 0$. Then, the transversal fluctuations of a maximal path should be of order $m^{-\zeta}$ with $\zeta = \zeta(\gamma)$ decreasing as γ decreases, up to some point where ζ reaches the value 0 (at which point a maximal path has transversal fluctuations of

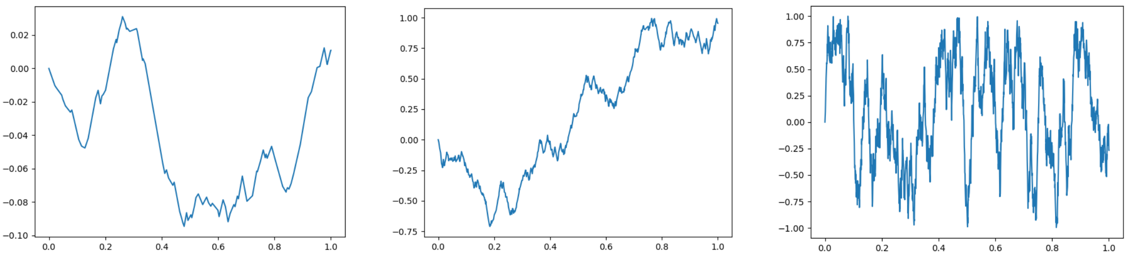


FIGURE 1. Simulation of H$_\gamma$-LPP with $m = 10^4$ (t, x, A all set to 1). The plots represent a maximizing path: from left to right, $\gamma = 1$ ($\mathcal{L}_m = 99$ in the picture, $m^{1/2} = 100$); $\gamma = 1/2$ ($\mathcal{L}_m = 510$, $m^{2/3} \approx 464$); $\gamma = 1/4$ ($\mathcal{L}_m = 1722$, $m^{4/5} \approx 1585$). We stress that the scale is different in all three plots, and we see that the transversal fluctuations are much smaller than 1 in the first case, and of order 1 in the second and third case.

order 1, see Figure 1 for an illustration). As discussed below (see in particular Section 3.3-(b)), it is natural to conjecture that $\zeta = (1-5\kappa/3) \vee 0 = \frac{\gamma-2/3}{1+\gamma} \vee 0$: transversal fluctuations should be much smaller than 1 when $\gamma > 2/3$ ($\kappa < 3/5$) and of order 1 when $\gamma < 2/3$ ($\kappa > 3/5$).

Remark 2.2. One could naturally generalize Hölder LPP to a *cone-shaped* LPP: one can define a region $\mathcal{R} = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}, f_2(t) \leqslant x \leqslant f_1(t)\}$, with $f_1 \leqslant f_2$ two functions $\mathbb{R}_+ \to \mathbb{R}$, and let the compatibility condition for Δ be that for any $(t_{i-1}, x_{i-1}), (t_i, x_i) \in \Delta$ we have $(t_i - t_{i-1}, x_i - x_{i-1}) \in \mathcal{R}$ (*i.e.* the next point in Δ has to be in the cone-shaped region $\mathcal{R}$ from the previous point). In this framework, H$_\gamma$-LPP is simply the *cone-shaped* LPP with $\mathcal{R} = \{(t,x), -t^\gamma \leqslant x \leqslant t^\gamma\}$, and one could easily adapt the proof of Theorem 2.1: the key quantity is $V(u) = \int_0^u |f_1 - f_2|(v)dv$, the area of $\mathcal{R}$ close to the origin, and one finds that $\mathcal{L}_m$ is of the order of $V^{-1}(1/m)$ (recovering the $m^{1/(1+\gamma)}$ in the Hölder case).

2.2. **Global Entropy constraint.** Another type of constraint that is natural to consider is a *global* constraint: we talk about an entropy constraint, since it arises naturally when considering random walk paths (the entropy being a measure of the non-likelihood of a path). This is a generalization of the study initiated in [6], which was motivated by applications to directed polymer in random heavy-tail environment and helped answer Conjecture 1.7 in [12] —we refer to Section 4 for an overview of how E-LPP can be applied. For any $a \geqslant b \geqslant 0$, $a > 0$, we define the (a,b)-Entropy of a set $\Delta = (t_i, x_i)_{1 \leqslant i \leqslant k}$ (again, the points are ordered $t_1 < \cdots < t_k$, and we use the convention $(t_0, x_0) = (0,0)$)

$$\mathrm{Ent}_{a,b}(\Delta) := \sum_{i=1}^{k} \frac{|x_i - x_{i-1}|^a}{|t_i - t_{i-1}|^b}\,. \tag{2.6}$$

In particular, we will be interested in two special subcases. First, when $b > 0$ and $a = b+1$: in that case, we can generalize the notion of entropy to continuous paths $s : [0,t] \to \mathbb{R}$, by $\mathrm{Ent}_b(s) = \int_0^t |s'(u)|^b du$, corresponding to the L^b norm of s' (it is related to the $(1,b)$-Sobolev norm of s) and the entropy of a set Δ corresponds to the entropy of the linear interpolation of Δ. Second, when $b = 0$: then the entropy can also be generalized to non-necessarily continuous paths $s : [0,t] \to \mathbb{R}$, by $\mathrm{Ent}_a(s) = \sup\big\{\sum_i |s(t_i) - s(t_{i-1})|^a\big\}$, the supremum being over all finite subdivisions $t_1 < \cdots < t_k$ of $[0,t]$. This corresponds to the "a-variation" norm of s (when $a = 1$ this is the total variation, and when $a = 2$ this is the quadratic variation). Note also that, considering $b > 0$ and $a = b/\gamma$ in (2.6), we have that

$$\Big(\mathrm{Ent}_{b/\gamma,b}(\Delta)\Big)^{\gamma/b} = \Big(\sum_{i=1}^{k} \frac{|x_i - x_{i-1}|^{b/\gamma}}{|t_i - t_{i-1}|^{b}}\Big)^{\gamma/b} \xrightarrow[b\to\infty]{} \sup_{1\leqslant i\leqslant k} \frac{|x_i - x_{i-1}|}{|t_i - t_{i-1}|^{\gamma}},$$

so that when $b \to \infty$ we formally recover the γ-Hölder norm of Δ (2.1).

Then, for some fixed $B > 0$, we define a compatibility set

$$\mathcal{E}_{a,b}^{B} := \Big\{\Delta \subset \mathbb{R}_+ \times \mathbb{R}\,;\, \Delta \text{ directed}, \mathrm{Ent}_{a,b}(\Delta) \leqslant B\Big\}, \tag{2.7}$$

so that a set of points is compatible if it can be collected by a path with entropy smaller than B. We then consider the Entropy constrained LPP, abbreviated as E-LPP, as

$$\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x}) := \sup\Big\{|\Delta|\,;\, \Delta \subset \Upsilon_m, \Delta \in \mathcal{E}_{a,b}^B\Big\}. \tag{2.8}$$

We prove the following result. (Again, we could define a point-to-point version of the E-LPP, by adding the condition that $(t,0) \in \Delta$: an analogous result would then hold for the point-to-point E-LPP.)

Theorem 2.3. *There are constants c_3, c_4 (depending only on a, b) such that for any t, x and any B, for any $1 \leqslant k \leqslant m$*

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x}) \geqslant k\Big) \leqslant \Big(\frac{c_3 (Bt^b/x^a)^{1/a} m}{k^{(a+b+1)/a}}\Big)^k, \tag{2.9}$$

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x}) \leqslant k\Big) \leqslant \exp\Big\{c_4 k\Big(1 - c_4\Big(\frac{(Bt^b/x^a)^{1/a}}{k^{(a+b)/a}} \wedge 1\Big)\frac{m}{k}\Big)\Big\}. \tag{2.10}$$

As a consequence, there is a constant C such that for any fixed t, x, a, b, B, $\mathbb{P}$-a.s. there is some m_0 such that

$$\frac{1}{C} \leqslant \frac{\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x})}{(Bt^b/x^a)^{1/(a+b+1)} m^{a/(a+b+1)}} \leqslant C \qquad \text{for all } m \geqslant m_0.$$

Again, the constants are uniform in the different parameters (and explicit, see the proof of Theorem 2.3), and this fact reveals to be very useful, in particular for the applications developed in Section 4.1.

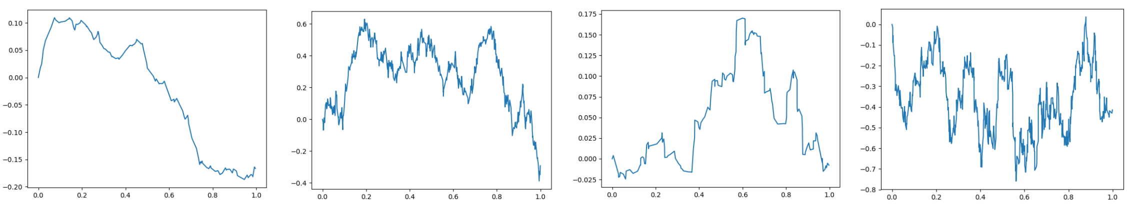


FIGURE 2. Simulation of E-LPP with $m = 10^4$ (t, x, B all set to 1), via a simulated annealing procedure (using a Glauber dynamic on paths, with transitions between paths differing by at most 1 point). The plots represents a path which collects a number of points that approximate $\mathcal{L}_m$, with different parameters a, b: from left to right, $a = 2, b = 1$ ($\mathcal{L}_m = 117$, $m^{1/2} = 100$), $a = 4, b = 1$ ($\mathcal{L}_m = 547$, $m^{2/3} \approx 464$), $a = 1, b = 0$ ($\mathcal{L}_m = 158$, $m^{1/2} = 100$), $a = 2, b = 0$ ($\mathcal{L}_m = 712$, $m^{2/3} \approx 464$). Again, we stress that the scale is different in all four plots – much smaller than 1 in the first and third, and of order 1 in the second and forth.

Also here, $\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)} = \mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x})$ is of order m^κ with $\kappa = a/(a+b+1)$, and it is natural to expect that $\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}/m^\kappa$ converges a.s. to a constant as $m \to \infty$. This convergence is discussed in Section 2.3, and the value of the constant in Appendix A. Notice that in the case where $a = b+1$ (which is one of the most natural, since it arises from LDP of random walks, see Remark 2.4), we find $\kappa = 1/2$, exactly as in the case of a Lipschitz constraint. In the case $b = 0$, we find $\kappa = a/(a+1)$ so $\kappa = 1/2$ when $a = 1$ (total variation case) and $\kappa = 2/3$ when $a = 2$ (quadratic variation case). As far as the transversal fluctuations of a maximal path are concerned, we argue in point (b) of Section 3.3 that it should be of order $m^{-\zeta}$, with $\zeta = (1-5\kappa/3) \vee 0 = \frac{b+1-2a/3}{a+b+1} \vee 0$: transversal fluctuations should be much smaller than 1 for $\kappa < 3/5$, and reach order 1 for $\kappa > 3/5$. See Figure 2 for an illustration.

Remark 2.4. Let us stress here that the entropy of a set Δ as defined in (2.6) appears naturally when considering large deviations for random walks: consider S a symmetric random walk with unbounded jumps, with stretch exponential tail $\mathbf{P}(S_1 = x) \overset{x\to\infty}{\sim} e^{-|x|^\nu}$, for some $\nu > 0$ (one may consider that $\nu = \infty$ includes the case of the usual simple random walk). Then, when considering the probability that a point (n, x_n) (with $n \to \infty$, $x_n \gg \sqrt{n}$) is visited (or collected) by the simple random walk path, we realize that

$$(2.11)\qquad -\log \mathbf{P}(S_n = x_n) \overset{n\to\infty}{\sim} \begin{cases} nI(x_n/n) & \text{if } \nu > 1, \text{ or } \nu \in (0,1) \text{ and } x_n \ll n^{1/(2-\nu)}, \\ J(x_n) & \text{if } \nu \in (0,1) \text{ and } x_n \gg n^{1/(2-\nu)}, \end{cases}$$

with some LDP rate functions $I(\cdot), J(\cdot)$. More specifically, we have $I(x) \sim x^2/2$ as $x \to 0$ (moderate deviation regime, see [10] for the standard Cramér case, [22] for the case $\nu \in (0,1)$), $I(x) = x^\nu$ as $x \to \infty$ (super-large deviation, one-jump deviation, see [23, Thm. 2.1]), and $J(x) = x^\nu$ (one-jump deviation, see [23, Thm. 2.1]). As such, the entropy defined in (2.6) is the natural scaling limit of the log-probability that a random walk path visits a given set of points. We chose the specific form (2.6) instead of using general LDP rate functions $I(\cdot), J(\cdot)$ because: (i) we are able to perform computations with this formula, (ii) we can usually bound the rate function $c|x|^a \leqslant I(x) \leqslant c'|x|^a$ for some $a > 0$. In (2.11), we therefore have: in the first part $a = 2, b = 1$ if $x_n/n \to 0$ or $a = \nu, b = \nu - 1$ ($\nu > 1$) if $x_n/n \to \infty$; in the second part, $a = \nu, b = 0$. However we keep the parameters a, b in the definition (2.6), to be able to deal with all these cases at once.

Remark 2.5. Let us stress here that we have a comparison between the Hölder and Entropy LPP: indeed, we observe that for $\Lambda \subset [0,t] \times \mathbb{R}$, we have $\mathcal{H}_\gamma^A \subset \mathcal{E}_{a,b}^B$ with $\gamma = (1+b)/a$ and $B = A^a t$. This is due to the fact that for any $\Delta = \{(t_i, x_i)\}_{1 \leqslant i \leqslant k}$ with $\mathrm{H}_\gamma(\Delta) \leqslant A$, we get that, using $\gamma = (1+b)/a$

$$\mathrm{Ent}_{a,b}(\Delta) = \sum_{i=1}^k \frac{|x_i - x_{i-1}|^a}{|t_i - t_{i-1}|^b} \leqslant \sum_{i=1}^k A^a |t_i - t_{i-1}|^{a\gamma - b} \leqslant A^a t\,.$$

This gives that $\mathcal{L}_m^{(\mathcal{E}_{a,b}^{A^a t})}(\Lambda) \geqslant \mathcal{L}_m^{(\mathcal{H}_{(1+b)/a}^A)}(\Lambda)$. On the other hand, it is not possible to get the other bound simply by comparison between local and global constraints.

2.3. Poissonian (point-to-point) version of path-constrained LPP.

Similarly to the standard LPP, we can define a Poissonian (point-to-point) version of the path constrained LPP, reproducing the idea of Hammersley [13] to prove the convergence of $\mathcal{L}_m/\sqrt{m}$.

For any $\lambda > 0$, let Υ_λ be a Poisson point process of intensity λ on $\mathbb{R}^2$, and we define the point-to-point version of path constrained LPPs. Let us consider $z = (x, y) \in \mathbb{R}^2$. For a given set $\Delta \subset \mathbb{R} \times (0, y)$, we set $\Delta^{(z)} = \Delta \cup \{z\}$ so that it extends Δ to make it end at z.

In the directed case, for any $t > 0$ we consider the domain $\Lambda_t = [0,t] \times \mathbb{R}$, and we consider the end-point (t, tu), for $u \in \mathbb{R}$. For any $A > 0$, $B > 0$, we define

$$\mathcal{L}^{(\mathcal{H}^A_\gamma)}_{\Upsilon_\lambda}(t,tu) = \mathcal{L}^{(\mathcal{H}^A_\gamma)}_{\lambda}(t,tu) := \sup\Big\{|\Delta|; \Delta \subset \Upsilon_\lambda \cap \Lambda_t, \Delta \text{ directed}, \mathrm{H}_\gamma(\Delta^{(t,tu)}) \leqslant A\Big\},$$
$$\mathcal{L}^{(\mathcal{E}^B_{a,b})}_{\Upsilon_\lambda}(t,zt) = \mathcal{L}^{(\mathcal{E}^B_{a,b})}_{\lambda}(t,tu) := \sup\Big\{|\Delta|; \Delta \subset \Upsilon_\lambda \cap \Lambda_t, \Delta \text{ directed}, \mathrm{Ent}_{a,b}(\Delta^{(t,tu)}) \leqslant Bt\Big\}.$$

Let us note that the entropy constraint grows linearly in t. We realize that in both cases, $\big(\mathcal{L}^{(\mathcal{C})}_\lambda(n,un)\big)_{n\geqslant 1}$ forms a super-additive ergodic sequence, in the sense that

$$\mathcal{L}^{(\mathcal{C})}_{\Upsilon_\lambda}(n+\ell,(n+\ell)u) \geqslant \mathcal{L}^{(\mathcal{C})}_{\Upsilon_\lambda}(n,nu) + \mathcal{L}^{(\mathcal{C})}_{\theta^n_u\Upsilon_\lambda}(\ell,\ell u), \tag{2.12}$$

where θ^n_u is the translation operator: $(t,x) \in \theta^n_u\Upsilon_\lambda$ if and only if $(t+n, x+un) \in \Upsilon_\lambda$. The super-additivity comes from the fact that the concatenation of two sets have: (i) a H_γ norm equal to the maximum of the H_γ norms of the two sets; (ii) an entropy equal to the sum of the entropies of the two sets. Therefore, Kingman's sub-additive ergodic theorem [18] implies the existence of the limit $\lim\limits_{t\to\infty, t\in\mathbb{N}} \frac{1}{t}\mathcal{L}^{(\mathcal{C})}_{\Upsilon_\lambda}(t,tu)$. In the following result we extend this limit to the continuous parameter $t \in \mathbb{R}_+$ and we show that it is finite.

Proposition 2.6. *For any $u \in \mathbb{R}$ and any $\lambda > 0$, the limits*

$$\mathtt{C}^{\mathrm{H}}_{\lambda,A}(u) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}^{(\mathcal{H}^A_\gamma)}_\lambda(t,tu), \qquad \mathtt{C}^{\mathrm{E}}_{\lambda,B}(u) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}^{(\mathcal{E}^B_{a,b})}_\lambda(t,tu) \tag{2.13}$$

exist a.s. and in L^1, and are finite, constant $\mathbb{P}$-a.s.

Moreover the constants $\mathtt{C}^{\mathrm{H}}_{\lambda,A}(u)$ and $\mathtt{C}^{\mathrm{E}}_{\lambda,B}(u)$ satisfy the following scaling relations

$$\mathtt{C}^{\mathrm{H}}_{\lambda,A}(u) = (\lambda A)^{\frac{1}{1+\gamma}}\, \mathtt{C}^{\mathrm{H}}_{1,1}\big(u\lambda^{\frac{1-\gamma}{1+\gamma}}A^{-\frac{2}{1+\gamma}}\big)\ ;\ \mathtt{C}^{\mathrm{E}}_{\lambda,B}(u) = (\lambda B^{1/a})^{\frac{a}{a+b+1}}\, \mathtt{C}^{\mathrm{E}}_{1,1}\big(u\lambda^{\frac{a-b+1}{a+b+1}}B^{-\frac{2}{a+b+1}}\big). \tag{2.14}$$

Proof. We start by proving (2.13). We have already noted that the super-additivity (2.12) gives directly the result for the limit along the integers $n \to \infty$. We can extend the limit along the real line $t \to \infty$, using that $t \mapsto \mathcal{L}^{(\mathcal{C})}_\lambda(t,tu)$ is non-decreasing.

It remains to prove that the constants are finite. We show how this is a consequence of our Theorems 2.1-2.3. Let us deal only with the Hölder case, and let us set $A = 1, \lambda = 1$ for simplicity. Thanks to (2.16), we get that $\mathcal{L}_1(t,tu) \overset{(d)}{=} \mathcal{L}_{t^{1+\gamma}}(1, t^{1-\gamma}u)$, therefore to prove that the constant in (2.13) is finite it suffices to show that $\limsup_{\rho\to\infty} \rho^{-1/(1+\gamma)}\mathcal{L}_\rho(1,\rho^{(1-\gamma)/(1+\gamma)}u) < +\infty$ a.s. For this purpose, removing the constraint we get that $\mathcal{L}_\rho(1,\rho^{(1-\gamma)/(1+\gamma)}u) \leqslant \mathcal{L}^{(\mathcal{H}^A_\gamma)}_{\Upsilon_\rho}(\Lambda_{1,\infty})$, where $\mathcal{L}^{(\mathcal{H}^A_\gamma)}_{\Upsilon_\rho}(\Lambda_{1,\infty})$ is the H_γ-LPP in the domain $\Lambda_{1,\infty} = [0,1]\times\mathbb{R}$ with a set Υ_ρ which is a Poisson point process of intensity ρ, see Section 2.1. We cannot directly apply Theorem 2.1 because $\Lambda_{1,\infty}$ is not bounded and Υ_ρ does not have a fixed number of points. However, we can write $\mathcal{L}^{(\mathcal{H}^A_\gamma)}_{\Upsilon_\rho}(\Lambda_{1,\infty}) = \lim_{j\to\infty}\mathcal{L}^{(\mathcal{H}^A_\gamma)}_{\Upsilon_\rho}(\Lambda_{1,j})$ with $\Lambda_{1,j} = [0,1]\times[-j,j]$, so that for any $v > 0$

$$\mathbb{P}\big(\mathcal{L}^{(\mathcal{H}^A_\gamma)}_{\Upsilon_\rho}(\Lambda_{1,\infty}) \geqslant v\rho^{1/(1+\gamma)}\big) = \lim_{j\to\infty}\mathbb{P}\big(\mathcal{L}^{(\mathcal{H}^A_\gamma)}_{\Upsilon_\rho}(\Lambda_{1,j}) \geqslant v\rho^{1/(1+\gamma)}\big).$$

Then, we denote $N_j^{(\rho)} := |\Upsilon_\rho \cap \Lambda_{1,j}|$ the number of Poisson points in $\Lambda_{1,j}$ ($\Lambda_{1,j}$ has volume $2j$). Then, using Theorem 2.1 (with $m = 4\rho j$), we can write

$$\begin{aligned} \mathbb{P}\big(\mathcal{L}_{\Upsilon_\rho}^{(\mathcal{H}_\gamma^A)}(\Lambda_{1,j}) \geqslant v\rho^{1/(1+\gamma)}\big) &\leqslant \mathbb{P}\big(N_j^{(\rho)} \geqslant 4\rho j\big) + \mathbb{P}\big(\mathcal{L}_{4\rho j}(\Lambda_{1,j}) \geqslant v\rho^{1/(1+\gamma)}\big) \\ &\leqslant \mathbb{P}\big(N_j^{(\rho)} \geqslant 4\rho j\big) + \Big(\frac{4c_1}{v^{1+\gamma}}\Big)^{v\rho^{1/(1+\gamma)}} . \end{aligned} \tag{2.15}$$

The first probability goes to 0 as $j \to \infty$ ($N_j^{(\rho)}$ is a Poisson random variable of parameter $2\rho j$), so that choosing $v_0 = (8c_1)^{1/(1+\gamma)}$, we obtain that

$$\mathbb{P}\big(\mathcal{L}_{\Upsilon_\rho}^{(\mathcal{H}_\gamma^A)}(\Lambda_{1,\infty}) \geqslant v_0\rho^{1/(1+\gamma)}\big) \leqslant 2^{-v_0\rho^{1/(1+\gamma)}} ,$$

which concludes the argument.

To show the scaling relation (2.14), we consider two different scaling relations satisfied by $\mathcal{L}_\lambda^{(\mathcal{H}_\gamma^A)}$ and $\mathcal{L}_\lambda^{(\mathcal{E}_{a,b}^B)}$. For this purpose, we start by considering the following maps:

(i) $(t,x) \mapsto (\lambda^{1/(1+\gamma)}t, \lambda^{\gamma/(1+\gamma)}x)$, which does not change the γ-Hölder norm of a set Δ;

(ii) $(t,x) \mapsto (\lambda^{a/(a+b+1)}t, \lambda^{(b+1)/(a+b+1)}x)$, which multiplies the entropy of a set Δ (and t) by $\lambda^{a/(a+b+1)}$.

Therefore, since the image of Υ_λ through these maps has the distribution of Υ_1, we obtain the following identities in distribution

$$\begin{aligned} &\mathcal{L}_\lambda^{(\mathcal{H}_\gamma^A)}(t,tu) \overset{(d)}{=} \mathcal{L}_1^{(\mathcal{H}_\gamma^A)}\big(\lambda^{1/(1+\gamma)}t, \lambda^{\gamma/(1+\gamma)}tu\big) \\ \text{and}\quad &\mathcal{L}_\lambda^{(\mathcal{E}_{a,b}^B)}(t,tu) \overset{(d)}{=} \mathcal{L}_1^{(\mathcal{E}_{a,b}^B)}\big(\lambda^{a/(a+b+1)}t, \lambda^{(b+1)/(a+b+1)}tu\big) . \end{aligned} \tag{2.16}$$

As a consequence, by using (2.13), we also get the existence of the following limits, for any fixed $t > 0$ and $u \in \mathbb{R}$, $A, B > 0$

$$\begin{aligned} &\lim_{\lambda\to\infty} \frac{1}{\lambda^{1/(1+\gamma)}} \mathcal{L}_\lambda^{(\mathcal{H}_\gamma^A)}\big(t, tu\lambda^{(1-\gamma)/(1+\gamma)}\big) = t\mathtt{C}^{\mathrm{H}}_{1,A}(u); \\ &\lim_{\lambda\to\infty} \frac{1}{\lambda^{a/(a+b+1)}} \mathcal{L}_\lambda^{(\mathcal{E}_{a,b}^B)}\big(t, tu\lambda^{(a-(b+1))/(a+b+1)}\big) = t\mathtt{C}^{\mathrm{E}}_{1,B}(u) . \end{aligned} \tag{2.17}$$

Note that we recover the same order for $\mathcal{L}_\lambda$ as in Theorems 2.1-2.3. Note also that the end-point has to be scaled with λ, except when $\gamma = 1$ or $a = b + 1$.

From (2.17) we directly obtain that

$$\begin{aligned} &\mathtt{C}^{\mathrm{H}}_{\lambda,A}(u) = \lambda^{1/(1+\gamma)}\mathtt{C}^{\mathrm{H}}_{1,A}(u\lambda^{-(1-\gamma)/(1+\gamma)}) , \\ \text{and}\qquad &\mathtt{C}^{\mathrm{E}}_{\lambda,B}(u) = \lambda^{a/(a+b+1)}\mathtt{C}^{\mathrm{E}}_{1,B}(u\lambda^{(b+1-a)/(a+b+1)}). \end{aligned} \tag{2.18}$$

Applying another scaling, we can also reduce to the case where $A = 1$, $B = 1$. We consider the following maps, that preserves the distribution of Υ_λ:

(i) $(t,x) \mapsto (A^{1/(1+\gamma)}t, A^{-1/(1+\gamma)}x)$, which divides the γ-Hölder norm by A;

(ii) $(t,x) \mapsto (B^{1/(a+b+1)}t, B^{-1/(a+b+1)}x)$, which multiplies the entropy by $B^{-1} \times B^{1/(a+b+1)}$.

Then, we obtain that

$$\begin{aligned} &\mathcal{L}_\lambda^{(\mathcal{H}_\gamma^A)}(t,tu) \overset{(d)}{=} \mathcal{L}_\lambda^{(\mathcal{H}_\gamma^1)}\big(A^{1/(1+\gamma)}t, A^{-1/(1+\gamma)}tu\big) \\ \text{and}\quad &\mathcal{L}_\lambda^{(\mathcal{E}_{a,b}^B)}(t,tu) \overset{(d)}{=} \mathcal{L}_\lambda^{(\mathcal{E}_{a,b}^1)}\big(B^{1/(a+b+1)}t, B^{-1/(a+b+1)}tu\big) . \end{aligned}$$

As a consequence, we have that

$$\mathtt{C}^{\mathrm{H}}_{1,A}(u) = A^{1/(1+\gamma)}\mathtt{C}^{\mathrm{H}}_{1,1}(u/A^{2/(1+\gamma)}),\quad \text{and}\quad \mathtt{C}^{\mathrm{E}}_{1,B}(u) = B^{1/(a+b+1)}\mathtt{C}^{\mathrm{E}}_{1,1}(u/B^{2/(a+b+1)}), \tag{2.19}$$

so that (2.18) and (2.19) give (2.14). □

Remark 2.7. When considering $t = 1, u = 0$ with $\lambda = m$, this corresponds to considering the LPP problem for paths $s : [0,1] \to \mathbb{R}$ in a Poisson point process of intensity m. In principle, one could therefore use (2.17) (with $\lambda = m$), together with a de-Poissonization argument (cf. [13]), in order to prove the convergence for the point-to-point version of the H_γ-LPP and E-LPP of Sections 2.1-2.2 to the constant on the r.h.s. of (2.17). We do not pursue in this direction, since it would not bring any technical novelty or much insight on the problem. We refer to Section 3.3-(a) for further discussion on the value of the constant. Let us stress that the argument should fail (and the constants differ) when the transversal fluctuations of the optimal path are of order 1 as discussed below Theorem 2.1: indeed, restricting the paths to stay in a box $[0,1] \times [-1,1]$ is then an important constraint.

2.4. **Discrete version of the directed path constrained LPP.** For the previous LPP models, we were considering the case of a continuous domain $\Lambda \subset \mathbb{R}^2$, and a set of points Υ that have a continuous distribution. Our idea is that these models can be thought as limits of discrete models, where Λ is a lattice domain, and Υ is a set of point on this domain. This is what is done in [6] in the directed random polymer context, where the E-LPP is considered both in the discrete and in the continuous setting and where it is the main tool to prove the convergence of the discrete model to a continuum limit —this was conjectured in [12].

Here below we briefly develop the discrete LPP setting. We let $n, h \in \mathbb{N}$, and we consider the (discrete) domain $\Lambda_{n,h} = [\![1,n]\!] \times [\![-h,h]\!]$. For $1 \leqslant m \leqslant \mathrm{Card}(\Lambda_{n,h})$, we consider Υ_m a set of m distinct points in $\Lambda_{n,h}$, chosen uniformly at random. Note that for $\Delta \subset \Lambda_{n,h}$, the definition of H_γ-Hölder norm (2.1) and entropy (2.6) of the set Δ still holds. We denote $L_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{n,h})$ and $L_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{n,h})$ the discrete analogues of the H-LPP and E-LPP: we then have results analogous to Theorems 2.1-2.3.

Theorem 2.8. *For any $n, h \geqslant 1$, and any $1 \leqslant k \leqslant m \leqslant 2nh$, we have that*

$$\mathbb{P}\Big(L_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{n,h}) \geqslant k\Big) \leqslant \Big(\frac{CAn^\gamma m}{hk^{1+\gamma}}\Big)^k, \quad \mathbb{P}\Big(L_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{n,h}) \leqslant k\Big) \leqslant e^{ck\big(1-c\big(\frac{An^\gamma}{hk^\gamma}\wedge 1\big)\frac{m}{k}\big)};$$

$$\mathbb{P}\Big(L_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{n,h}) \geqslant k\Big) \leqslant \Big(\frac{CB^{1/a}n^{b/a}m}{hk^{(a+b+1)/a}}\Big)^k, \quad \mathbb{P}\Big(L_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{n,h}) \leqslant k\Big) \leqslant e^{ck\big(1-c\big(\frac{B^{1/a}n^{b/a}}{hk^{(a+b)/a}}\wedge 1\big)\frac{m}{k}\big)}.$$

We recover with this result that in the discrete setting: (i) $L_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{n,h})$ is of order $(An^\gamma/h)^{1/(1+\gamma)}m^{1/(1+\gamma)}$; (ii) $L_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{n,h})$ is of order $(Bn^b/h^a)^{1/(a+b+1)}m^{a/(a+b+1)}$. The proof of Theorem 2.8 is identical to those of its continuous counterparts Theorems 2.1-2.3 (see for instance the proof of [6, Theorem 3.1-(ii)]), and we leave it to the reader.

3. Non-directed LPP

Let us now develop the fact that the notion of compatibility allows for even more general constraints, and for example enables us to deal with non-directed paths. To do so, we consider a natural framework: we work with a time horizon $[0,t]$, and define the γ-Hölder norm and the Entropy of a subset $\Delta = (x_i)_{1\leqslant i\leqslant k}$ of $\mathbb{R}^2$ (the points are considered in a given order), by considering the optimal γ-Hölder norm or Entropy of a path going through the points of Δ (in the correct order) in a time horizon t:

$$\mathbf{H}_\gamma(t,\Delta) := \inf\left\{ \sup_{1\leqslant i\leqslant k} \frac{\|x_i - x_{i-1}\|}{|t_i - t_{i-1}|^\gamma} \,;\, t_1 < \cdots < t_k \text{ subdivision of } [0,t] \right\}, \tag{3.1}$$

$$\mathbf{Ent}_{a,b}(t,\Delta) := \inf\left\{ \sum_{i=1}^{k} \frac{\|x_i - x_{i-1}\|^a}{|t_i - t_{i-1}|^b} \,;\, t_1 < \cdots < t_k \text{ subdivision of } [0,t] \right\}, \tag{3.2}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$. Another way of presenting it is by saying that $\mathbf{H}_\gamma(t,\Delta)$ (resp. $\mathbf{Ent}_{a,b}(t,\Delta)$) is smaller than A if and only if there exists a path $s:[0,t]\to\mathbb{R}^2$ collecting the points of Δ which has γ-Hölder norm (resp. Entropy) smaller than A.

Here again, the case $b>0$ with $a=b+1$ will be of particular interest for us, since it arises naturally from a LDP for non-directed random walks to visit a certain set of points (*i.e.* considering the probability that there are some times $t_1<\cdots<t_k$ such that $S_{t_i}=x_i$). It can be extended to continuous curves $s:[0,t]\to\mathbb{R}^2$, or more precisely, to their traces $\varrho=\{s(u), u\in[0,t]\}$, by taking the infimum of $\int_0^t \|\tilde{s}'(u)\|^a du$ over all possible parametrization $\tilde{s}:[0,t]\to\mathbb{R}^2$ of ϱ. The case $b=0$ arises also when considering random walks with increments with a stretch-exponential tail, and correspond to the a-variation norm of a curve $s:[0,t]\to\mathbb{R}$ (which does not depend on the parametrization of the curve).

Let us notice right away that we are able to identify the optimal subdivision $0\leqslant t_1<\cdots<t_k\leqslant t$ used by a path to collect all points of Δ:

- For the Hölder case (3.1), we find that the optimal choice for the subdivision is $t_i-t_{i-1}=t\|x_i-x_{i-1}\|^{1/\gamma}\big(\sum_{i=1}^k\|x_i-x_{i-1}\|^{1/\gamma}\big)^{-1}$ (so that all terms in the sup are equal). Then we obtain that the γ-Hölder norm of Δ is

$$\mathbf{H}_\gamma(t,\Delta) = \frac{1}{t^\gamma}\Big(\sum_{i=1}^k \|x_i-x_{i-1}\|^{1/\gamma}\Big)^\gamma . \tag{3.3}$$

  We note that when $\gamma=1$, the definition (2.1) corresponds to the total length of the linear interpolation of the points of Δ, and can therefore be extended to continuous curves $s:[0,t]\to\mathbb{R}^2$, by $\int_0^t\|s'(u)\|du$, the total length of the curve. It does not depend on the parametrization but only on the trace $\varrho=\{s(u),u\in[0,t]\}$.
- For the Entropy case (3.2), we find that the optimal choice for the subdivision is $t_i-t_{i-1}=t\|x_i-x_{i-1}\|^{a/(b+1)}\big(\sum_{i=1}^k\|x_i-x_{i-1}\|^{a/(b+1)}\big)^{-1}$ – note that when $a=b+1$, t_i-t_{i-1} is just proportional to the distance between the points. Then we obtain that the Entropy of Δ is

$$\mathbf{Ent}_{a,b}(t,\Delta) = \frac{1}{t^b}\Big(\sum_{i=1}^k \|x_i-x_{i-1}\|^{a/(b+1)}\Big)^{b+1} . \tag{3.4}$$

  Note that when $a=b+1$, (3.4) corresponds to the $(b+1)$-th power of the length of the linear interpolation of the points of Δ.

Remark 3.1. In view of (3.3)-(3.4) (and the comments below), we see that the H_γ-LPP and the E-LPP are equivalent. We indeed have $\mathbf{Ent}_{a,b}(t,\Delta)=t\mathbf{H}_\gamma(t,\Delta)^a$ with $\gamma=(b+1)/a$, or also $\mathbf{H}_\gamma(t,\Delta)=t^{-\gamma}\mathbf{Ent}_{a,b}(t,\Delta)^\gamma$ with $b=0$ and $a=1/\gamma$. Hence, we will focus simply on the non-directed E-LPP, since the Entropy and Hölder constraints are easily related to each other.

We will work with the domain $\Lambda_r = \{x \in \mathbb{R}^2, \|x\| \leqslant r\}$, the disk of radius r (for symmetry reasons, but this choice is not crucial). For $m \geqslant 1$, Υ_m is a set of m independent variables uniform in Λ_r. Then, for some fixed $B > 0$, we define the non-directed *Entropy compatible* sets with time horizon $[0, t]$,

$$\mathscr{E}_{a,b}^{t,B} = \left\{ \Delta \subset \mathbb{R}^2 \, ; \, \mathbf{Ent}_{a,b}(t, \Delta) \leqslant B \right\},$$

and finally the *non-directed* LPP,

$$\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r) := \sup \Big\{ |\Delta| \, ; \, \Delta \subset \Upsilon_m, \Delta \in \mathscr{E}_{a,b}^{B}(t) \Big\}. \tag{3.5}$$

(We use a curly font for $\mathscr{L}$ and $\mathscr{E}$ to visually mark the difference with the directed LPPs.) We prove the following result, for non-directed LPP.

Theorem 3.2. *There exist constants c_5, c_6 such that for any t, r and B, for any $1 \leqslant k \leqslant m$*

$$\mathbb{P}\Big(\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r) \geqslant k \Big) \leqslant \Big(\frac{c_5 (B t^b / r^a)^{2/a} m}{k^{2(b+1)/a}} \Big)^k , \tag{3.6}$$

$$\mathbb{P}\Big(\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r) \leqslant k \Big) \leqslant e^{-c_6 m} + \exp\Big\{ c_6 k \Big(1 - c_6 \frac{m^{a/2(b+1)}}{k} \big(B t^b / r^a \big)^{1/(b+1)} \Big) \Big\} . \tag{3.7}$$

Finally, there is some $C > 0$ such that $\mathbb{P}$-a.s. there is some m_0 such that

$$\frac{1}{C} \leqslant \frac{\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r)}{m \wedge (B t^b / r^a)^{\frac{1}{b+1}} m^{\frac{a}{2(b+1)}}} \leqslant C \qquad \text{for all } m \geqslant m_0 . \tag{3.8}$$

Also here, the constants in (3.6)-(3.7) are uniform in the parameters m, B, t, r, allowing for a dependence of these parameters on m.

In view of Remark 3.1 above, we obtain an analogous statement for non-directed H_γ-LPP (take $b = 0$, $a = 1/\gamma$, and $B = t A^{1/\gamma}$ in Theorem 3.2). For instance, the last statement of Theorem 3.2 can be read (with obvious notations) as

$$\mathbb{P} - a.s. \; \exists m_0 > 0 \colon \quad \frac{1}{C} \leqslant \frac{\mathscr{L}_m^{(\mathscr{H}_\gamma^{t,A})}(\Lambda_r)}{(A t^\gamma / r)^{1/\gamma} \, m^{1/(2\gamma)}} \leqslant C \quad \text{for all } m \geqslant m_0 . \tag{3.9}$$

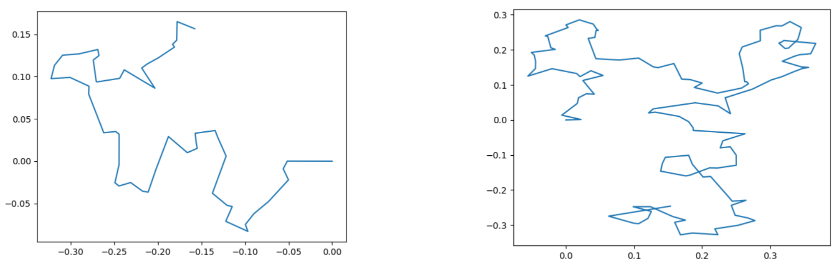


FIGURE 3. Simulation of non-directed LPP with $m = 10^3$ in $[-0.5, 0.5]^2$ (t, B set to 1), via a simulated annealing procedure (using a Glauber dynamic on paths, with transitions between paths differing by at most one point). The plots represents a path which collects a number of points that approximates $\mathcal{L}_m$ with different values for $\gamma = (b+1)/a$: on the left, $\gamma = 1$ ($\mathcal{L}_m = 53$ in the picture, $m^{1/2} \approx 32$); $\gamma = 3/4$ ($\mathcal{L}_m = 128$, $m^{2/3} = 100$). Note that the scale is different in the two plots – quite smaller than 1 in the first case, of order 1 in the second case.

We have that $\mathscr{L}_m$ is of order m^κ with $\kappa = \frac{a}{2(b+1)} \wedge 1$ (or $\kappa = \frac{1}{2\gamma} \wedge 1$), and it is also natural to expect that $\mathscr{L}_m / m^\kappa$ converges a.s. to a constant as $m \to \infty$. We highlight the fact that, in the non-directed case, we find that $\kappa = 1/2$ (as for the standard LPP), both for an entropy constraint with $a = b+1$ (the standard case when considering entropy arising from LDP of random walks) and for a Lipschitz constraint ($\gamma = 1$, corresponding to a length constraint, see discussion after (3.3)).

3.1. **Poissonian version of the model.** In the non-directed framework, we are also able to define a Poissonian version of the model. For any $z \in \mathbb{R}^2$ and any $r > 0$, we will consider sets Δ and extend them to end at rz (we denote $\Delta^{(rz)}$ this extension), in order to define a point-to-point version (and use sub-additivity techniques). The main difference with the directed case is that we need here to decide what is the time horizon t_r to reach that point. As further discussed below, the only reasonable choice is to pick $t_r = r^{1/\gamma}$, resp. $r^{a/(b+1)}$, which is the time needed to reach rz with H_γ norm of order 1, resp. with entropy of order t_r. We will also see that the models present some interest only when $\gamma = 1$ (the H_γ norm is then just the length of the path) or when $a = b+1$ (and the entropy derives from standard LDP).

For any $A > 0$, $B > 0$, we define

$$\mathscr{L}_{\Upsilon_\lambda}^{(\mathscr{H}_\gamma^A)}(rz) = \mathscr{L}_{\lambda}^{(\mathscr{H}_\gamma^A)}(rz) := \sup\left\{|\Delta|; \Delta \subset \Upsilon_\lambda, \mathbf{H}_\gamma(t_r, \Delta^{(rz)}) \leqslant A\right\},$$
$$\mathscr{L}_{\Upsilon_\lambda}^{(\mathscr{E}_{a,b}^B)}(rz) = \mathscr{L}_{\lambda}^{(\mathscr{E}_{a,b}^B)}(rz) := \sup\left\{|\Delta|; \Delta \subset \Upsilon_\lambda, \mathbf{Ent}_{a,b}(t_r, \Delta^{(rz)}) \leqslant Bt_r\right\}.$$

Let us realize right away that the two models are equivalent (on the contrary to Section 3 where the dependence on t was different for the two models, cf. Remark 2.5): (i) from (3.3), having $\mathbf{H}_\gamma(t_r, \Delta) \leqslant A$ is equivalent to $\sum_{i=1}^k \|x_i - x_{i-1}\|^{1/\gamma} \leqslant A^{1/\gamma} t_r$; (ii) from (3.4), having $\mathbf{Ent}_{a,b}(t_r, \Delta) \leqslant Bt_r$ is equivalent to $\sum_{i=1}^k \|x_i - x_{i-1}\|^{a/(b+1)} \leqslant B^{1/(b+1)} t_r$. We therefore focus only on the Entropy case – we set $\gamma = (b+1)/a$, and drop the super-script $\mathscr{E}_{a,b}^B$ to ease the notations.

In order for the sequence $\big(\mathscr{L}_\lambda(nz)\big)_{n\geqslant 1}$ to be super-additive ergodic (*i.e.* verify (2.12)), we need to have $\gamma \leqslant 1$ so that $t_r + t_s \leqslant t_{r+s}$ for any $r, s \in \mathbb{R}_+$ (using that $(t_r + t_s)^\gamma \leqslant t_r^\gamma + t_s^\gamma$ for $\gamma \leqslant 1$). Indeed, super-additivity simply comes from the above remark that $\mathbf{Ent}_{a,b}(t_{r+s}, \Delta) \leqslant Bt_r$ is equivalent to $\sum_{i=1}^k \|x_i - x_{i-1}\|^{1/\gamma} \leqslant B^{1/(b+1)} t_{r+s}$, together with $t_r + t_s \leqslant t_{r+s}$. This gives, as for (2.13), the following convergence (a.s. and in L^1),

$$\mathbf{C}_{\lambda,B}(z) = \lim_{r\to\infty} \frac{1}{r} \mathscr{L}_\lambda(rz). \tag{3.10}$$

Note that by symmetry, the constant $\mathbf{C}_{\lambda,B}(z)$ depends only on $\|z\|$.

Additionally, $\mathscr{L}_\lambda(rz)$ verifies some scaling relations. Note that here, in view of the definition (3.4), we need to scale both coordinates in the same way: we use the map $x \mapsto \lambda^{1/2} x$, which preserves the condition $\mathbf{Ent}_{a,b}(t_r, \Delta^{(r)}) \leqslant Bt_r$ thanks to our choice of $t_r = r^{1/\gamma}$ – this is crucial here, and is the main reason for our choice of time horizon. The image of Υ_λ though this map has the distribution of Υ_1, so we obtain that

$$\mathscr{L}_\lambda(rz) \overset{(d)}{=} \mathscr{L}_1\big(\lambda^{1/2} rz\big).$$

As a consequence of this scaling relation and (3.10), for any $r > 0$, we have the convergence

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^{1/2}} \mathscr{L}_\lambda(rz) = r\mathbf{C}_{1,B}(z). \tag{3.11}$$

Note that we recover the correct order for $\mathscr{L}_\lambda$ only when $a = b+1$ or $\gamma = 1$ cf. (3.8) and (3.9), (in which case the time horizon is $t_r = r$), but not in other cases. This is due to the constraint that rz has to be visited in the time horizon $t_r = r^{1/\gamma}$: when $\gamma < 1$, it somehow stretches the paths, which cannot wander as much as in the "free" case. An idea to overcome this problem would be to consider the "free end-point" version of this non-directed Poisson LPP in some time horizon t – however preventing from the use of super-additivity. Then, the natural question would be to determine the typical end-to-end distance.

3.2. **Discrete version.** Here again, we can define a discrete version of the model (we do it only in the entropy case), by considering the discrete domain $\Lambda_d = \{x \in \mathbb{Z}^2, \|x\| \leqslant d\}$. Then, for $m \leqslant \mathrm{Card}(\Lambda_d)$ we consider Υ_m a set of m distinct points of Λ_d, chosen uniformly at random. Here we consider a discrete time horizon n, and we slightly modify the definition of the entropy of a set $\Delta \subset \Lambda_d$ compared to (3.2), to fit the discrete setting:

$$\mathbf{Ent}_{a,b}(n,\Delta) := \inf\left\{ \sum_{i=1}^{k} \frac{\|x_i - x_{i-1}\|^a}{|n_i - n_{i-1}|^b}\,;\, n_1 < \cdots < n_k \;\text{ subdivision of } [\![1,n]\!] \right\}. \tag{3.12}$$

Then, we define $L_m^{(\mathscr{E}_{a,b}^{n,B})}(\Lambda_d)$ the corresponding non-directed E-LPP, and have a result analogous to Theorem 3.2 (we display here only the analogous of (3.6)).

Theorem 3.3. *We have a constant such that for any n, d, and any $1 \leqslant k \leqslant m \leqslant |\Lambda_d|$,*

$$\mathbb{P}\Big(L_m^{(\mathscr{E}_{a,b}^{n,B})}(\Lambda_d) \geqslant k \Big) \leqslant \Big(\frac{C (B n^b / r^2)^{1/a} m}{k^{2(b+1)/a}} \Big)^k .$$

The proof of Theorem 3.3 is identical to those of its continuous counterparts Theorem 3.2, and we leave it to the reader.

3.3. **Open questions and directions.** Our main goal here has been to introduce a generalized Last Passage Percolation, and the results we present here give the first properties of such models, which are already useful in some contexts, see the two potential applications we develop in Section 4 below. However, many questions are raised, and we provide here a few important open problems that remain – some of them seem out of reach for the moment.

(a) Show the convergence of the LPP. We have shown that the H_γ-LPP, E-LPP or non-directed LPP, generically denoted $\mathcal{L}_m$, are of order m^κ for some $\kappa > 0$. What we did not prove but strongly believe is that $\mathcal{L}_m/m^\kappa$ converges (a.s. and in L^1) to a constant $\mathtt{C}$ as $m \to \infty$ (cf. Remark 2.7). The next step would then be to identify this constant, or the constant $\mathtt{C}_{1,1}(0) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}(t,0)$ in the Poissonian setting (cf. (2.13)), and its dependence on the parameters of the model (in particular in γ or a, b, since the dependence in t, A, B can be derived thanks to scaling arguments, see Proposition 2.6, (2.14)). In Appendix A.1, we present some simulations for the directed H_γ-LPP in Poissonian environment of Section 2.3 (with A and λ set to 1), and we display a graph of the constant $\mathtt{C}_{1,1}(0)$ as a function of γ, see Figure 6. Another natural question is also to determine the dependence in the end-point of the constants $\mathtt{C}_{1,1}(u)$ appearing in (2.13).

In Appendix A.2 we present some simulations for the directed E-LPP, which do not allow us to make some convincing predictions.

(b) Once the constant $\mathtt{C}_{1,1}(0) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}(t,0)$ has been determined, the next natural step is to identify the fluctuations of $\mathcal{L}(t,0)$ around $t\,\mathtt{C}_{1,1}(0)$. The question is to know whether there is an analogue of Theorem 1.1 to the generalized LPP. As far as the directed setting is concerned, simulations presented in Appendix A suggest that the model is still in the KPZ universality class. It is reasonable to believe that in the Poisson setting of Section 2.3 (setting $\lambda = 1$, A or B equal to 1, and $u = 0$) the convergence in (2.13) should generalize to the following convergence in distribution

$$\frac{\mathcal{L}_1(t,0) - t\,\mathtt{C}_{1,1}(0)}{t^{1/3}} \xrightarrow{(d)} F_{GUE}\,. \tag{3.13}$$

(The dependence on γ or on a, b is hidden in the constant $\mathtt{C}_{1,1}(0)$ and possibly in the normalization of F_{GUE}.) According to Remark 1.2, it is also natural to expect that the typical transversal fluctuations of a maximal path should be of order $t^{2/3}$.

Applying the map $(t,x) \mapsto (m^{-\kappa}t, m^{\kappa-1}x)$ as done in Section 2.3 (with $\kappa = 1/(1+\gamma)$ or $\kappa = a/(a+b+1)$), which preserve the constraints but multiplies the intensity of the Poisson point process by m, the convergence in (3.13) above (with $t = m^{\kappa}$) transforms to

$$\frac{\mathcal{L}_m(1,0) - \mathtt{C}_{1,1}(0)m^{\kappa}}{m^{\kappa/3}} \xrightarrow{(d)} F_{GUE}\,. \tag{3.14}$$

As far as the transversal fluctuations of a maximal path are concerned, the transformation above suggest that they are of order $m^{\kappa-1} \times (m^{\kappa})^{2/3} = m^{5\kappa/3-1}$.

When we consider the directed H_γ-LPP or E-LPP of Sections 2.1-2.2, in which we draw m points uniformly in a domain $\Lambda_{t,x}$ rather than a Poisson point process of intensity $m/(2tx)$ (recall $|\Lambda_{t,x}| = 2tx$), the relation above tells us that the transversal fluctuations of a maximal path should be of order $m^{-\zeta}$ with $\zeta = (1-5\kappa/3) \vee 0$. In the case $\kappa > 3/5$ the path is "blocked" by the border of the domain $\Lambda_{t,x}$, and oscillates much more inside the domain. This should make the constant $\mathtt{C} = \lim_{m\to\infty} \mathcal{L}_m/m^{\kappa}$ different than the corresponding $\mathtt{C}_{1,1}(0) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}(t,0)$ in that case.

(c) As far as the non-directed setting is concerned, the above discussion is even more far-reaching: because of its "directedness", the point-to-point Poissonian version seems useless here to prove that the limit $\mathtt{C} = \lim_{n\to\infty} \mathscr{L}_m/m^{\kappa}$ (with $\kappa = \frac{a}{2(b+1)}$ or $\kappa = \frac{1}{2\gamma}$) exists, even if we believe it does exist. We did not perform simulations to test the value of $\mathtt{C}$ and its dependence on the parameters a, b or γ, because of the high computation time even for a small number of points m. It is still reasonable to believe that the model is also in the KPZ universality class, that is $m^{-\kappa/3}(\mathscr{L}_m - \mathtt{C}m^{\kappa})$ converges in distribution to F_{GUE}, and that typical transversal fluctuations for the model are of order $m^{-\zeta}$ with $\zeta = (1-5\kappa/3) \vee 0$.

4. Some applications of the (entropy) path-constrained LPP

We now present two applications of the directed and non-directed LPPs, to the context of polymer models.

4.1. Application I: a model for a directed polymer in Poissonian environment.

We define here a very natural variational problem, which encapsulate the energy-entropy competition inherent to models of polymers in random environment. The random environment is given by a Poisson point process Υ_λ on $\mathbb{R}_+ \times \mathbb{R}$ of intensity $\lambda > 0$ (its law is denoted $\mathbb{P}$), and for $\beta > 0$, we define the following (point to point) variational problem

$$\mathscr{Z}_{\lambda,\beta}(t) := \sup_{s:[0,t]\to\mathbb{R}, s(0)=s(t)=0} \Big\{ \beta\,\big|s \cap \Upsilon_\lambda\big| - \mathrm{Ent}(s) \Big\}\,, \tag{4.1}$$

with $\mathrm{Ent}(s)$ defined as in (2.6) – because Υ_λ is countable, $\mathrm{Ent}(s)$ is well-defined. Here, $|s \cap \Upsilon_\lambda|$ the number of points collected by the path, is viewed as a measure of the energy of a trajectory s, so this variational problem constitute a simplified model to study the energy-entropy competition of polymer models. Again, the central cases that we have in mind is when $a = b+1$ or $b = 0$ in the definition of the entropy (2.6), see Remark 2.4 (when the entropy derives from the LDP of a simple random walk, we have $a = 2, b = 1$). The idea of this model is similar to that of [9] which considers a Brownian polymer in Poissonian medium. However, here, we somehow consider only the ground states, that is trajectories maximizing the energy-entropy balance, and we also allow for more general entropy than that of the Brownian motion (for which $a = 2, b = 1$).

Let us stress that the variational problem (4.1) has already appeared in [3] (in the case $a = 2, b = 1$) as a solution for a Hamilton-Jacobi equation used to study the stationary solutions of a Burgers equation (with a forcing induced by the points of a Poisson Point Process). It has also proven to be useful for the study of the thermodynamic limit for directed polymers, see [4].

First of all, we notice that as in Section 2.3, $\mathcal{Z}_{\lambda,\beta}(t)$ is a super-additive ergodic sequence – the entropy of the concatenation of two paths is the sum of the entropies of the two paths –, so that Kingman's sub-additive ergodic theorem gives that the limit

$$f(\lambda,\beta) := \lim_{t\to\infty} \frac{1}{t}\mathcal{Z}_{\lambda,\beta}(t) \tag{4.2}$$

exists a.s. and in L^1, and is $\mathbb{P}$-a.s. constant. The fact that there exists a constant c such that $\mathbb{P}$ a.s. $\limsup \frac{1}{t}\mathcal{Z}_{\lambda,\beta}(t) \leqslant c$ (so that the constant $f(\lambda,\beta)$ is finite) derives from our estimates in Theorem 2.3: the scheme of proof is identical to that of Proposition 4.1 below (together with the argument in Section 2.3, see (2.15)), so we skip it – we mention that this fact was an important part of the study in [3]. We also have scaling relations for $\mathcal{Z}_{\lambda,\beta}(t)$. Indeed, consider the two following maps: (i) $(t,x) \mapsto (\lambda^{-a/(a+b)}t, \lambda^{-b/(a+b)}x)$ whose image of Υ_λ has distribution Υ_1 and which preserves the entropy; (ii) $(t,x) \mapsto (\beta^{-1/(a+b)}t, \beta^{1/(a+b)}x)$, which multiplies the entropy by β, while preserving the distribution of Υ_λ. We therefore obtain that

$$\mathcal{Z}_{\lambda,\beta}(t) \overset{(d)}{=} \mathcal{Z}_{1,\beta}\big(\lambda^{-a/(a+b)}t\big) \qquad \text{and} \qquad \mathcal{Z}_{\lambda,\beta}(t) \overset{(d)}{=} \beta\mathcal{Z}_{\lambda,1}\big(\beta^{-1/(a+b)}t\big)\,. \tag{4.3}$$

A first consequence is that we get that $f(\lambda,\beta) = (\beta^{a+b+1}\lambda^a)^{1/(a+b)}f(1,1)$, where $f(1,1)$ is a constant that needs to be determined. Another consequence is that, if we consider the alternative problem where we take $\lambda\to\infty$ (instead of $t\to\infty$), we get that, for any fixed positive t,β, the limit

$$\lim_{\lambda\to+\infty} \frac{1}{\lambda^{a/(a+b)}}\mathcal{Z}_{\lambda,\beta}(t) = tf(1,\beta) = t\beta^{(a+b+1)/(a+b)}f(1,1) \tag{4.4}$$

exists a.s. and in L^1.

We considered the Poissonian point-to-point version for the sake of simplicity (in particular to be able to use scaling relations), but one could naturally define a "m-points" version of the model. More precisely, considering the domain $\Lambda_{1,1} = [0,1]\times[-1,1]$, and Υ_m a set of m points taken uniformly and independently in $\Lambda_{1,1}$, we can define the variational problem, for $\beta > 0$

$$Z_{m,\beta} := \sup_{s:[0,1]\to[-1,1]} \Big\{\beta|s\cap\Upsilon_m| - \mathrm{Ent}(s)\Big\}\,. \tag{4.5}$$

Then, in view of (4.4), we expect that a "de-Poissonization" technique would enable us to show that there is a constant $\mathtt{Cst} > 0$ such that

$$\lim_{m\to\infty} \frac{1}{m^{a/(a+b)}} Z_{m,\beta} = \beta^{(a+b+1)/(a+b)}\mathtt{Cst}\,. \tag{4.6}$$

(Since $\Lambda_{1,1}$ has volume 2, we have an intensity of points $\lambda = m/2$, so we expect that $\texttt{Cst} = 2^{a/(a+b)} f(1,1)$.) In the most standard case $a = 2, b = 1$ (deriving from LDP of the simple random walk), we therefore find that the variational problem $Z_{m,\beta}$ is of order $\beta^{4/3} m^{2/3}$ – this is much larger than $\sqrt{m}$ which is the order when we consider the case of a uniformly bounded entropy.

We stress that one can easily find the correct order for $Z_{m,\beta}$ thanks to the results of Section 2.2. Indeed, we can write from (4.5) that

$$Z_{m,\beta} = \sup_{B \geqslant 0} \Big\{ \beta \sup_{s:[0,1]\to\mathbb{R}, \mathrm{Ent}(s)=B} \{|s \cap \Upsilon_m|\} - B \Big\} . \tag{4.7}$$

Then, since in Theorem 2.3 it is proven that

$$\sup_{s, \mathrm{Ent}(s) \leqslant B} \{|s \cap \Upsilon_m|\} \asymp B^{1/(a+b+1)} m^{a/(a+b+1)} \wedge m,$$

one readily sees that the maximum in (4.7) is attained for (and is of the order of) $B \asymp (\beta^{a+b+1} m^a)^{1/(a+b)} \wedge (\beta m)$.

We can actually make this precise, and prove deviation bounds for $Z_{m,\beta}$.

Proposition 4.1. *There are constants c_7, c_8, and some K_0 (depending only on a, b) such that for any $K > K_0$, and provided that m is large enough so that $(\beta m^a)^{1/(a+b)} \leqslant m$, we have*

$$\mathbb{P}\Big(Z_{m,\beta} \geqslant K (\beta^{a+b+1} m^a)^{1/(a+b)} \Big) \leqslant e^{-c_7 K (\beta m^a)^{1/(a+b)}} , \tag{4.8}$$

$$\mathbb{P}\Big(Z_{m,\beta} \leqslant \frac{1}{K} (\beta^{a+b+1} m^a)^{1/(a+b)} \Big) \leqslant e^{-c_8 K^{b/a} (\beta m^a)^{1/(a+b)}} \tag{4.9}$$

As a consequence, there is some $C > 0$ such that for any fixed $\beta > 0$, $\mathbb{P}$-a.s. there is some m_0 such that

$$\frac{1}{C} \leqslant \frac{Z_{m,\beta}}{(\beta^{a+b+1} m^a)^{1/(a+b)} \wedge (\beta m)} \leqslant C , \qquad \textit{for all } m \geqslant m_0 .$$

Perspectives. For this model, some important questions remain unanswered:

(i) what is the value of the constant $\texttt{Cst}$ (or equivalently of the constant $f(1,1)$)? In view of (4.7), and since we believe that $\sup_{\mathrm{Ent}(s)\leqslant B} \{|s \cap \Upsilon_m|\} \sim B^{1/(a+b+1)} \texttt{C} m^{\kappa}$ with $\kappa = 1/(a+b+1)$ and $\texttt{C}$ (we used the scaling of relation (2.14) that should hold also in the non-Poissonian case, that is, $\texttt{C}_B = B^{1/(a+b+1)} \texttt{C}$), we conjecture that the supremum in (4.7) is equal to

$$\mathbf{c}_{a,b} (\beta \texttt{C} m^{\kappa})^{(a+b+1)/(a+b)}, \quad \text{where} \quad \mathbf{c}_{a,b} = (a+b)(a+b+1)^{-(a+b+1)/(a+b)}$$

(the supremum is attained for $B = (\beta \texttt{C} m^{\kappa}/(a+b+1))^{(a+b+1)/(a+b)}$). Since the simulations of Appendix A suggest that the constant $\texttt{C}$ is equal to 1, we can therefore conjecture that

$$\lim_{m\to\infty} \frac{Z_{m,\beta}}{(\beta^{a+b+1} m^a)^{1/(a+b)}} = \mathbf{c}_{a,b} .$$

(ii) what does the maximizer of $\mathcal{Z}_{\lambda,\beta}(t)$ (or $Z_{m,\beta}$) look like? for example what is its typical transversal fluctuation exponent? We mention that in [3, 4], the results are mostly qualitative, such as the existence and coalescence of semi-infinite maximizers for this model.

We believe that this model deserves further investigation, and would lead to a better understanding of the energy-entropy balance in polymer models, and improve our understanding of the Burgers equation with stationary forcing.

Proof of Proposition 4.1. The proof is a relatively simple application of Theorem 2.3, and makes use of the fact that the estimates (2.9)-(2.10) are uniform in the parameters.

• For the upper bound, we use the idea sketched above: for any $v > 0$, we decompose the variational problem by writing

$$Z_{m,\beta} \leqslant \Big(\beta \sup_{s,\mathrm{Ent}(s)\in[0,v]} \big\{|s \cap \Upsilon_m|\big\}\Big) \vee \sup_{k\geqslant 1} \Big(\beta \sup_{s,\mathrm{Ent}(s)\in[2^{k-1}v,2^k v]} \big\{|s \cap \Upsilon_m|\big\} - 2^{k-1}v\Big)\,.$$

Hence, a union bound gives that

$$\mathbb{P}\big(Z_{m,\beta} \geqslant v\big) \leqslant \mathbb{P}\Big(\sup_{\mathrm{Ent}(s)\leqslant v} \big\{|s \cap \Upsilon_m|\big\} \geqslant v/\beta\Big) + \sum_{k=1}^{\infty} \mathbb{P}\Big(\sup_{\mathrm{Ent}(s)\leqslant 2^k v} \big\{|s \cap \Upsilon_m|\big\} \geqslant 2^{k-1}v/\beta\Big).$$

Since $\sup_{\mathrm{Ent}(s)\leqslant 2^k v} \big\{|s\cap\Upsilon_m|\big\} \leqslant \mathcal{L}_m^{(\mathcal{E}_{a,b}^{2^k v})}$, we use Theorem 2.3-(2.9) with $v = K(\beta^{a+b+1}m^a)^{1/(a+b)}$, and we obtain that provided that K is large enough,

$$\begin{aligned}
&\mathbb{P}\big(Z_{m,\beta} \geqslant K(\beta^{a+b+1}m^a)^{1/(a+b)}\big)\\
&\qquad\leqslant \Big(c_3 K^{-(a+b)/a}\Big)^{-K(\beta m^a)^{1/(a+b)}} + \sum_{k=1}^{\infty} \Big(c(2^k K)^{-(a+b)/a}\Big)^{-2^k K(\beta m^a)^{1/(a+b)}}\\
&\qquad\leqslant c\exp\Big(-K(\beta m^a)^{1/(a+b)}\Big)\,.
\end{aligned}$$

• For the lower bound, this is easier: for any $v > 0$, we have that

$$Z_{m,\beta} \geqslant \beta \sup_{s,\mathrm{Ent}(s)\leqslant v} \big\{|s \cap \Upsilon_m|\big\} - v\,.$$

With $v := (2K)^{-1}(\beta^{a+b+1}m^a)^{1/(a+b)}$, we obtain that

$$\begin{aligned}
\mathbb{P}\Big(Z_{m,\beta} \leqslant \frac{1}{K}(\beta^{a+b+1}m^a)^{1/(a+b)}\Big) &\leqslant \mathbb{P}\Big(\sup_{s,\mathrm{Ent}(s)\leqslant v} \big\{|s \cap \Upsilon_m|\big\} \leqslant v/\beta\Big)\\
&\leqslant \exp\Big(-cK^{-1}(\beta m^a)^{1/(a+b)} \times K^{(a+b)/a}\Big)\,,
\end{aligned}$$

where the last inequality comes from Theorem 2.3-(2.10), provided that K is large enough.

The almost sure statement holds thanks to the previous bounds, by an easy application of Borel-Cantelli lemma.

4.2. **Application II: (continuous) non-directed polymers in heavy-tail environment.** The directed E-LPP have already proved to be useful to understand the transversal fluctuations and scaling limits of directed polymers in heavy-tail random environment, see [6]. The continuous limit of the model is found to be an energy-entropy variational problem, and E-LPP appears central to ascertain its well-posedness. Here, we define an analogous variational problem in the non-directed setting, and show that it is well defined. It should also appear as the scaling limit of some non-directed polymer model in heavy-tail random environment – that we plan on studying more thoroughly.

As a continuum disorder field, we let $\mathcal{P} := \{(w_i, x_i, y_i) \colon i \geqslant 1\}$ be a Poisson Point Process on $[0,\infty)\times\mathbb{R}^2$, of intensity $\mu(\mathrm{d}w\mathrm{d}x\mathrm{d}y) = \frac{\alpha}{2} w^{-\alpha-1}\mathbb{1}_{\{w>0\}}\mathrm{d}w\mathrm{d}x\mathrm{d}y$ – it derives from the scaling of a discrete field of disorder with heavy-tail distribution. For a continuous path $s : [0,1] \to \mathbb{R}^2$, we can then define the continuum energy it collects by summing the weights in $\mathcal{P}$ "collected" by s (that is sitting on the trace of s), $\pi(s) = \sum_{(x_i,y_i)\in s} w_i$. We can also define its length $\ell(s) = \int_0^1 \|s'(u)\|\mathrm{d}u$, and we consider $\ell(s)^\nu$ for some $\nu > 1$ as a measure of its entropy. Indeed, if s is a linear interpolation of a finite number of points in $\mathcal{P}$, then

$\ell(s)^\nu$ is nothing but the non-directed E-LPP defined in (3.4) with $a = b+1$ and $b+1 = \nu$. This choice derives from LDP for a random walk, and $\nu = 2$ corresponds to the moderate deviation regime of the simple random walk.

Thanks to the non-directed LPP of Section 3, we are able to show that the energy/entropy variational problem is well defined, when $\alpha \in (2/\nu, 2)$.

Proposition 4.2. *For any $\nu > 1$, the following variational problem is well defined for all $\beta \geqslant 0$, when $\alpha \in (2/\nu, 2)$,*

$$\mathcal{T}_\beta^{(\nu)} := \sup_{\substack{s:[0,1]\to\mathbb{R}^2 \\ s(0)=0,\, \ell(s)<\infty}} \left\{ \beta\pi(s) - \ell(s)^\nu \right\}. \tag{4.10}$$

For $\beta > 0$, we have that $\mathcal{T}_\beta^{(\nu)} > 0$ a.s. and $\mathbb{E}[(\mathcal{T}_\beta^{(\nu)})^\kappa] < \infty$ for any $\kappa < \alpha - 2/\nu$. Moreover, for any $\alpha \in (2/\nu, 2)$, we have the scaling relation

$$\mathcal{T}_\beta^{(\nu)} \overset{(d)}{=} \beta^{\frac{\nu\alpha}{\nu\alpha-2}}\, \mathcal{T}_1^{(\nu)}. \tag{4.11}$$

On the other hand, if $\alpha \in (0, 2/\nu]$, we have that $\mathcal{T}_\beta^{(\nu)} = +\infty$ a.s.

Up to now, polymers in random environment have mostly been considered in the directed framework, see [8] for a thorough review, or in the semi-directed context of stretched polymers, see [15, 29], or [16] for a review. Proposition 4.2 therefore shows that our generalized LPP can be useful to study non-directed polymers: the variational problem can be thought as an energy/entropy model for a continuous polymer in continuous random environment. The main question remaining is then to describe what a maximizer of (4.10) look like.

Perspectives. The most natural question is now to consider a (discrete) non-directed polymer model in random environment (the Hamiltonian being the sum of the weights of the sites visited by the random walk), and prove its convergence to the variational problem of Proposition 4.2, in the case of a heavy-tail environment. More generally, the study of non-directed polymers in random environment is of great interest, and should be pursued.

Proof of Proposition 4.2. The proof is inspired by that in [6, Section 4]. We fix $\nu > 1$ in the following, so we drop it from the notation $\mathcal{T}_\beta^{(\nu)} =: \mathcal{T}_\beta$.

$*$ *Scaling relations.* For $\alpha \in (0,2)$ and $\rho > 0$ we consider $\varphi_\rho(w,x) := (\rho^{2/\alpha} w, \rho x)$ which scales space by ρ and weights by $\rho^{2/\alpha}$ respectively. For the Poisson point process $\mathcal{P}$ defined in Section 4.2, we get that for any $\rho > 0$, $\varphi_\rho(\mathcal{P}) \overset{(d)}{=} \mathcal{P}$. Then, applying this scaling with $\rho = \beta^{-\alpha/(\nu\alpha-2)}$ (if $\alpha \neq 2/\nu$) we obtain the following scaling relation for any $\beta > 0$

$$\mathcal{T}_\beta = \beta^{\frac{\nu\alpha}{\nu\alpha-2}} \sup_{s,\, \ell(s)<\infty} \left\{ \beta^{-2/(\nu\alpha-2)}\pi(s) - \left(\beta^{-\alpha/(\nu\alpha-2)}\ell(s)\right)^\nu \right\} \overset{(d)}{=} \beta^{\frac{\nu\alpha}{\nu\alpha-2}} \mathcal{T}_1\,. \tag{4.12}$$

$*$ *Positivity.* We show that for any $\beta > 0$, $\mathcal{T}_\beta > 0$. Moreover we show that a.s. $\mathcal{T}_\beta = +\infty$ if $\alpha \in (0, 2/\nu]$. For any $u > 0$, let us consider $\mathcal{D}_u := [0,\infty) \times [-u,u]^2$. We have

$$\mathcal{T}_\beta \geqslant \max_{(w,x,y)\in\mathcal{P}\cap\mathcal{D}_u} \{\, \beta\, w \,\} - (\sqrt{2}u)^\nu\,.$$

We observe that, by considering the ordered statistics of $\mathcal{P}\cap\mathcal{D}_u$ (see the proof of Lemma 4.3 below), we get that

$$\max_{(w,x,y)\in\mathcal{P}\cap\mathcal{D}_u} \{\, w \,\} \overset{\text{(d)}}{=} (2u)^{2/\alpha} X \qquad \text{with } X \overset{\text{(d)}}{=} \text{Exp}(1)^{-1/\alpha}.$$

Then, with $c = \beta^{-1}(2)^{\nu/2-2/\alpha}$, we obtain that

$$\mathbb{P}\Big(\mathcal{T}_\beta > 0\Big) \geqslant \lim_{u\to 0} \mathbb{P}\Big(X \geqslant cu^{\nu-2/\alpha}\Big) = 1, \quad \text{when}\, \alpha > 2/\nu\,,$$
$$\mathbb{P}\Big(\mathcal{T}_\beta = +\infty\Big) \geqslant \lim_{u\to\infty} \mathbb{P}\Big(X \geqslant cu^{\nu-2/\alpha}\Big) = 1, \quad \text{when}\, \alpha < 2/\nu\,.$$

For the case $\alpha = 2/\nu$ we consider the set $\mathcal{G}_u := [\beta^{-1}(4\sqrt{2}u)^{2/\alpha}, \infty) \times [u, 2u)^2$. As before we have that, on the event $\mathcal{P} \cap \mathcal{G}_u \neq \varnothing$,

$$\mathcal{T}_\beta \geqslant \max_{(w,x,y)\in\mathcal{P}\cap\mathcal{G}_u} \{\,\beta\, w\,\} - (2\sqrt{2}u)^\nu \geqslant (4\sqrt{2}u)^{2/\alpha} - (2\sqrt{2}u)^\nu = (2\sqrt{2}u)^{2/\alpha}\,. \tag{4.13}$$

Since $\varphi_{1/u}(\mathcal{P}) \overset{(\mathrm{d})}{=} \mathcal{P}$ we have that $\mathbb{P}(\mathcal{P} \cap \mathcal{G}_u \neq \varnothing) \geqslant c > 0$, with c independent of u. Therefore, since the events $(\{\mathcal{P} \cap \mathcal{G}_{2^k} \neq \varnothing\})_{k\in\mathbb{N}}$ are independent, the Borel-Cantelli lemma gives that infinitely many of them occur with probability 1, and (4.13) leads to conclude that a.s. $\mathcal{T}_\beta = +\infty$.

$*$ *Finite moments.* We define, for any interval $[c, d)$, the variational problem restricted to paths of length $\ell(s) \in [c, d)$:

$$\mathcal{T}\big([c, d)\big) := \sup_{s,\, \ell(s)\in[c,d)} \big\{\beta\pi(s) - \ell(s)^\nu\big\}\,. \tag{4.14}$$

Then, we can write that $\mathcal{T}_\beta = \mathcal{T}_\beta([0, 1)) \vee \sup_{k\geqslant 0} \mathcal{T}_\beta([2^k, 2^{k+1}))$, and observe that scaling space by $2^{-(k+1)}$ we obtain that

$$\begin{aligned}\mathcal{T}_\beta\big([2^k, 2^{k+1})\big) &\overset{(\mathrm{d})}{=} \sup_{s,\,\ell(s)\in[1/2,1)} \Big\{2^{(k+1)2/\alpha}\pi(s)\beta - 2^{(k+1)\nu}\ell(s)^\nu\Big\} \\ &\leqslant 2^{(k+1)2/\alpha}\beta \sup_{s,\,\ell(s)\leqslant 1} \pi(s) - 2^{k\nu}\,.\end{aligned}$$

Below, we show the following lemma.

Lemma 4.3. *For any $\alpha > 1/2$, and any $\upsilon < \alpha$, there is a constant c_υ such that for any $t > 1$ we have*

$$\mathbb{P}\Big(\sup_{s,\,\ell(s)\leqslant 1} \pi(s) > t\Big) \leqslant c_\upsilon t^{-\upsilon}\,. \tag{4.15}$$

Hence, for $\alpha > 1/2$ and $\upsilon < \alpha$, for any $t > 1 \wedge \beta$, we get by a union bound that

$$\begin{aligned}\mathbb{P}\big(\mathcal{T}_\beta > t\big) &\leqslant \mathbb{P}\big(\mathcal{T}_\beta([0, 1)) > t\big) + \sum_{k=0}^{+\infty} \mathbb{P}\big(\mathcal{T}_\beta([2^k, 2^{k+1})) > t\big) \\ &\leqslant \mathbb{P}\Big(\sup_{s,\,\ell(s)\leqslant 1} \pi(s) > t/\beta\Big) + \sum_{k=0}^{+\infty} \mathbb{P}\Big(\sup_{s,\,\ell(s)\leqslant 1} \pi(s) > \beta^{-1}2^{-(k+1)2/\alpha}(t + 2^{k\nu})\Big) \\ &\leqslant c_\upsilon\beta^\upsilon t^{-\upsilon} + c_\upsilon\beta^\upsilon \sum_{k=0}^{+\infty} 2^{2k\upsilon/\alpha}(t + 2^{k\nu})^{-\upsilon} \leqslant c'_\upsilon\beta^\upsilon t^{\upsilon\left(\frac{2}{\alpha\nu}-1\right)}\,.\end{aligned}$$

The last inequality holds by separating the terms $2^{k\nu} < t$ $(k \leqslant \frac{1}{\nu}\log_2 t)$ and $2^{k\nu} \geqslant t$ $(k \leqslant \frac{1}{\nu}\log_2 t)$ in the last sum. Since υ can be arbitrarily close to α, for any $\kappa < \alpha - 2/\nu$ we have that there exists a constant $c_\kappa = c_\kappa(\beta)$ such that for any $t \geqslant 1$

$$\mathbb{P}\big(\mathcal{T}_\beta > t\big) \leqslant c_\kappa t^{-\kappa}\,. \tag{4.16}$$

This concludes the proof that $\mathbb{E}[(\mathcal{T}_\beta)^\kappa] < \infty$ for any $\kappa < \alpha - 2/\nu$, and it only remains to prove Lemma 4.3.

Proof of Lemma 4.3. Since we consider the optimization problem with length smaller than 1, we can restrict the Poisson point process $\mathcal{P}$ to the disk $D_1 = \{x \in \mathbb{R}^2, \|x\| \leqslant 1\}$. We can then rewrite a realization of $\mathcal{P}$ using its ordered statistic $\mathcal{P} = (M_i, X_i)_{i\geqslant 1}$, where M_i is the i-th largest weight, and X_i its position. The distribution of $(M_i, X_i)_{i\geqslant 1}$ can be given as follows: $(M_i)_{i\geqslant 1}$ and $(X_i)_{i\geqslant 1}$ are independent, X_i are i.i.d. uniform in D_1, and $M_i = \pi^{1/\alpha}(E_1 + \cdots + E_i)^{-1/\alpha}$, where $(E_i)_{i\geqslant 1}$ are i.i.d. Exp(1) random variables.

Then, we have that $\pi(s) = \sum_{i=1}^{\infty} M_i \mathbb{1}_{\{X_i \in s\}}$, and using that M_i is non-decreasing, we get that

$$\pi(s) = \sum_{j=0}^{\infty} \sum_{i=2^j}^{2^{j+1}} M_i \mathbb{1}_{\{X_i \in s\}} \leqslant \sum_{j=0}^{\infty} M_{2^j} \mathscr{L}_{2^{j+1}} \,, \tag{4.17}$$

where $\mathscr{L}_m$ is the non-directed LPP defined in (3.5), with set of points $\Upsilon_m := \{X_1, \ldots, X_m\}$ (with $r = 1, t = 1, b = 0, a = 1, B = 1$). Now, we will use that $\mathscr{L}_i$ is of order $\sqrt{i}$ and M_i of order $i^{-1/\alpha}$. Since $\alpha < 2$, we can fix some $\delta > 0$ (small) such that $1/\alpha - 1/2 > 2\delta$, and by a union bound, we get that

$$\mathbb{P}\big(\sup_{s,\, \ell(s)\leqslant 1} \pi(s) > t \big) \leqslant \sum_{j=0}^{\infty} \mathbb{P}\Big(M_{2^j} \mathscr{L}_{2^{j+1}} > c_\delta\, t (2^j)^{1/2-1/\alpha+2\delta} \Big) \tag{4.18}$$

where $c_\delta = (\sum_{j\geqslant 0} (2^j)^{1/2-1/\alpha+2\delta})^{-1}$.

Then, we use Theorem 3.2-(3.6) to get that there is a constant c_0, independent of C, such that

$$\mathbb{P}\big(\mathscr{L}_{2^{j+1}} > C \log t (2^j)^{1/2+\delta} \big) \leqslant e^{-c_0 C \log t (2^j)^\delta} \leqslant t^{-c_0 C (2^j)^\delta} \,. \tag{4.19}$$

On the other hand, we also have that $i^{1/\alpha} M_i = \pi^{1/\alpha}\big((E_1 + \cdots + E_i)/i\big)^{-1/\alpha}$, so that $\mathbb{E}[(i^{1/\alpha} M_i)^{(1-\delta)\alpha}]$ is bounded by a constant that depends only on δ. Markov's inequality then gives that for any C'

$$\mathbb{P}\Big(M_{2^j} > C' \frac{t}{\log t} (2^j)^{-1/\alpha+\delta} \Big) \leqslant \frac{c}{C'^{(1-\delta)\alpha}} (\log t)^{(1-\delta)\alpha} t^{-(1-\delta)\alpha} (2^j)^{-\delta(1-\delta)\alpha} \,. \tag{4.20}$$

Combining (4.19)-(4.20), we get that

$$\begin{aligned} \mathbb{P}\Big(M_{2^j} \mathscr{L}_{2^{j+1}} > & c_\delta\, t (2^j)^{1/2-1/\alpha+2\delta} \Big) \\ & \leqslant \mathbb{P}\Big(\mathscr{L}_{2^{j+1}} > C \log t (2^j)^{1/2+\delta} \Big) + \mathbb{P}\Big(M_{2^j} > \frac{c_\delta}{C} \frac{t}{\log t} (2^j)^{-1/\alpha+\delta} \Big) \\ & \leqslant t^{-c_0 C (2^j)^\delta} + c''_\delta t^{-(1-2\delta)\alpha} (2^j)^{-\delta(1-\delta)\delta} \,, \end{aligned}$$

so that summing over j in (4.18), we get that

$$\mathbb{P}\Big(\sup_{s,\, \ell(s)\leqslant 1} \pi(s) > t \Big) \leqslant t^{-c'_0 C} + c'_\delta t^{-(1-2\delta)\alpha} \leqslant 2 c'_\delta t^{-(1-2\delta)\alpha} \,.$$

The last inequality holds provided that C has been fixed large enough. This concludes the proof, since δ is arbitrary. □

5. Proofs of the path-constrained LPP bounds

We prove here Theorems 2.1-2.3-3.2. The almost sure statements are straightforward applications of the first parts of the theorems (via the Borel-Cantelli lemma), so we skip their proof. The ideas are similar to those developed in [6, Part 1], in a special case of the E-LPP.

5.1. **Hölder-constrained LPP.** We prove first (2.4), and then (2.5).

Upper bound. Define $H_k(t,A)$ the set of k (ordered) elements up to time-horizon t that have a γ-Hölder norm bounded by A:

$$H_k(t,A) = \Big\{(t_i,x_i)_{1\leqslant i\leqslant k}\,;\, 0<t_1<\cdots<t_k<t, \mathrm{H}_\gamma\big((t_i,x_i)_{1\leqslant i\leqslant k}\big)\leqslant A\Big\}\,.$$

Then, we are able to compute exactly the volume of $\mathcal{H}_k(t,A)$.

Lemma 5.1. *For any $t>0$ and $A>0$, we have for any $k\geqslant 1$*

$$\mathrm{Vol}\big(H_k(t,A)\big) = (2A)^k \frac{\Gamma(1+\gamma)^k}{\Gamma\big(k(1+\gamma)+1\big)} t^{k(1+\gamma)}.$$

In particular, it gives that there exists some constant $C=C_\gamma\leqslant c(1+\gamma)^{-1/2}$ such that

$$\mathrm{Vol}(H_k(t,A)) \leqslant \Big(\frac{CAt^{1+\gamma}}{k^{1+\gamma}}\Big)^k.$$

Proof. The key to the computation is the induction formula below, based on the decomposition over the left-most point in $\mathcal{H}_k(t,A)$ at position (u,y) (by symmetry we can assume $y\geqslant 0$): it leaves $k-1$ points with remaining time horizon $t-u$:

$$\mathrm{Vol}\big(H_k(t,A)\big) = 2\int_{u=0}^t\int_{y=0}^{Au^\gamma}\mathrm{Vol}\big(H_{k-1}(t-u,A)\big)dydu = 2A\int_0^t u^\gamma \mathrm{Vol}\big(H_{k-1}(t-u,A)\big)du.$$

We give the details of the induction for the sake of completeness, but the proof is a straightforward calculation.

For $k=1$, the computation is easy:

$$\mathrm{Vol}(H_1(t,B)) = 2\int_{u=0}^t\int_{y=0}^{Au^\gamma} dudy = 2A\int_0^t u^\gamma du = \frac{2A}{1+\gamma}t^{1+\gamma}\,.$$

For $k\geqslant 2$, by induction, we have

$$\mathrm{Vol}\big(H_k(t,A)\big) = (2A)^k\frac{\Gamma(1+\gamma)^{k-1}}{\Gamma\big((k-1)(1+\gamma)+1\big)}\times\int_{u=0}^t u^\gamma(t-u)^{(k-1)(1+\gamma)}du.$$

Then, by a change of variable $w=u/t$, we get

$$\begin{aligned}\int_{u=0}^t u^\gamma(t-u)^{(k-1)(1+\gamma)}du &= t^{(k-1)(1+\gamma)+\gamma+1}\int_0^1 w^\gamma(1-w)^{(k-1)(1+\gamma)}dw\\ &= t^{k(1+\gamma)}\frac{\Gamma\big(\gamma+1\big)\Gamma\big((k-1)(1+\gamma)+1\big)}{\Gamma\big(k(1+\gamma)+1\big)}\,,\end{aligned}$$

and this completes the induction.

For the inequality in the second part of the lemma, we use Stirling's formula to get that for any $\alpha>0$, as $k\to\infty$ we have $\Gamma\big(k\alpha+1\big)\sim\sqrt{2\pi\alpha k}(k\alpha/e)^{k\alpha}$. Hence, with the formula for $\mathrm{Vol}(H_k(t,A))$, we end up with the bound

$$\mathrm{Vol}(H_k(t,A)) \leqslant \frac{c}{\sqrt{k(1+\gamma)}}\bigg(\frac{2A\Gamma(1+\gamma)t^{1+\gamma}}{\big((1+\gamma)/e\big)^{1+\gamma}k^{1+\gamma}}\bigg)^k \leqslant \bigg(\frac{c'At^{1+\gamma}}{(1+\gamma)^{1/2}k^{1+\gamma}}\bigg)^k, \tag{5.1}$$

where we used that $\Gamma(1+\gamma)\sim\sqrt{2\pi\gamma}(\gamma/e)^\gamma$ as $\gamma\to\infty$ for the last inequality. □

We then use this Lemma to control the probability that $\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x})$ is larger than some k:

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x}) \geqslant k\Big) = \mathbb{P}(\mathcal{N}_k \geqslant 1) \leqslant \mathbb{E}[\mathcal{N}_k], \tag{5.2}$$

where $\mathcal{N}_k = \mathrm{Card}\{\Delta \subset \Upsilon_m\,;\, \Delta \in \mathcal{H}_\gamma^A\}$ is the number of sets of k points in Υ_m that are $\mathcal{H}_\gamma^A$-compatible. Since all the points of $\Upsilon_m = \{Z_1, \dots, Z_m\}$ are exchangeable, we have

$$\mathbf{E}[\mathcal{N}_k] = \binom{m}{k} \mathbb{P}\Big(\exists\, \sigma \in \mathfrak{S}_k \text{ s.t. } (Z_{\sigma(1)}, \dots, Z_{\sigma(k)}) \in H_k(t, A)\Big)\,.$$

Since the $(Z_i)_{1\leqslant i\leqslant m}$ are i.i.d. uniform in $\Lambda_{t,x} = [0,t] \times [-x, x]$ (of volume $2tx$), we get that

$$\mathbb{E}[N_k] = \binom{m}{k} \times \frac{\mathrm{Vol}\big(H_k(t, A)\big)}{(2tx)^k/k!}, \tag{5.3}$$

where the $k!$ comes from the fact that we rearrange the Z_i's so that $0 < t_1 < \cdots < t_k < t$. Using Lemma 5.1 together with $\binom{m}{k} \leqslant m^k/k_1$, we therefore obtain that

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x}) \geqslant k\Big) \leqslant \Big(\frac{CAt^\gamma m}{xk^{1+\gamma}}\Big)^k. \tag{5.4}$$

This gives the upper bound (2.9).

Lower bound. For any $k \geqslant 1$, let us consider the following sub-boxes of $\Lambda_{t,x}$, for $1 \leqslant i \leqslant 4k$:

$$\mathcal{B}_i := \Big[\frac{(i-1)t}{4k}, \frac{it}{4k}\Big) \times \Big[-\frac{A(t/k)^\gamma}{2} \wedge x, \frac{A(t/k)^\gamma}{2} \wedge x\Big]\,.$$

Then, we realize that if there are at least k boxes among $\{\mathcal{B}_{2i}\}_{1\leqslant i\leqslant 2k}$ containing (at least) one point, then this set of k points has a γ-Hölder norm which is bounded by $A(k/t)^\gamma/(t/k)^\gamma \leqslant A$. Hence, we get that

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x}) \geqslant k\Big) \leqslant \mathbb{P}\Big(\sum_{i=1}^{2k} \mathbb{1}_{\{|\Upsilon_m \cap \mathcal{B}_{2i}| \geqslant 1\}} \leqslant k\Big) = \mathbb{P}\Big(\sum_{i=1}^{2k} \mathbb{1}_{\{|\Upsilon_m \cap \mathcal{B}_{2i}| = 0\}} \leqslant k\Big)\,. \tag{5.5}$$

For the last probability, we use a union bound and the fact that the $\mathbb{1}_{\{|\Upsilon_m \cap \mathcal{B}_{2i}|=0\}}$ are exchangeable, to get that

$$\begin{aligned} 1 - \mathbb{P}\Big(\sum_{i=1}^{2k} \mathbb{1}_{\{|\Upsilon_m \cap \mathcal{B}_{2i}|=0\}} \leqslant k\Big) &\leqslant \binom{2k}{k} \mathbb{P}\Big(\Upsilon_m \cap \bigcup_{i=1}^{k} \mathcal{B}_i = \varnothing\Big) \\ &\leqslant 2^{2k}\Big(1 - \frac{At^\gamma}{8k^\gamma x} \wedge \frac{1}{4}\Big)^m\,. \end{aligned} \tag{5.6}$$

In the second inequality we used that Υ_m is a set of m independent random variables uniform in $\Lambda_{t,x}$ (of volume $2tx$), and that $\bigcup_{i=1}^k \mathcal{B}_i$ has a volume of $(\frac14 At^{1+\gamma}k^{-\gamma}) \wedge \frac{tx}{2}$. Then, we use that $1 - x \leqslant e^{-x}$ for any x, to get that

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{H}_\gamma^A)}(\Lambda_{t,x}) \leqslant k\Big) \leqslant \exp\Big\{ck\Big(1 - c\Big(\frac{At^\gamma/x}{k^\gamma} \wedge 1\Big)\frac{m}{k}\Big)\Big\}\,,$$

which concludes the proof of the (2.5).

5.2. **Entropy-constrained LPP.** We prove first (2.9), and then (2.10). The proofs are analogous to that of the Hölder case (and to what is done in [6, Section 3]), we give the details for the sake of completeness.

Upper bound. Define $E_k(t,B)$ the set of k (ordered) elements up to time-horizon t that have an entropy bounded by B:

$$E_k(t,B) = \Big\{ (t_i,x_i)_{1\leqslant i\leqslant k}\,;\, 0<t_1<\cdots<t_k<t, \mathrm{Ent}_{a,b}\big((t_i,x_i)_{1\leqslant i\leqslant k}\big) \leqslant B \Big\}\,.$$

Then, analogously to Lemma 5.1, we are able to compute exactly the volume of $E_k(t,B)$.

Lemma 5.2. *For any $t>0$ and $B>0$, we have for any $k\geqslant 1$*

$$\mathrm{Vol}\big(E_k(t,B)\big) = 2^k \big(\tfrac{1}{a}\big)^k \frac{\Gamma(\frac{1}{a})^k}{\Gamma\big(\frac{k}{a}+1\big)} \frac{\Gamma(\frac{a+b}{a})^k}{\Gamma\big(k\frac{(a+b)}{a}+1\big)} \times B^{k/a} t^{k(a+b)/a}.$$

In particular, it gives that there exists some constant $C=C_{a,b}$ such that

$$\mathrm{Vol}(E_k(t,B)) \leqslant \Big(\frac{C B^{1/a} t^{(a+b)/a}}{k^{(a+b+1)/a}}\Big)^k\,.$$

During the course of the proof, one finds that $C_{a,b}\leqslant c(a+b)^{-1/2}$.

Proof. Again, using a decomposition over the left-most point in $E_k(t,B)$ at position (u,y) (by symmetry we can assume $y\geqslant 0$): it leaves $k-1$ points with remaining time horizon $t-u$ and constraint $B-\frac{|y|^a}{u^b}$, we obtain the key induction formula below

$$\mathrm{Vol}\big(E_k(t,B)\big) = 2\int_{u=0}^{t}\int_{y=0}^{(Bu^b)^{1/a}} \mathrm{Vol}\Big(E_{k-1}(t-u, B-\tfrac{y^a}{u^b})\Big) dy du.$$

We give the details of the induction for the sake of completeness, but the proof is a straightforward calculation (slightly more involved than that of the previous section).

First of all, we have for $k=1$

$$\mathrm{Vol}(E_1(t,B)) = 2\int_{u=0}^{t}\int_{y=0}^{(Bu^b)^{1/a}} du dy = 2B^{1/a}\int_0^t u^{b/a} du = 2B^{1/a}\frac{a}{a+b} t^{(a+b)/a}\,.$$

For $k\geqslant 2$, by induction, we have

$$\begin{aligned}\mathrm{Vol}\big(E_k(t,B)\big) &= 2^{k-1}\big(\tfrac{1}{a}\big)^{k-1} \frac{\Gamma(\frac{1}{a})^{k-1}}{\Gamma\big((k-1)/a+1\big)} \frac{\Gamma(\frac{a+b}{a})^{k-1}}{\Gamma\big((k-1)\frac{(a+b)}{a}+1\big)} \\ &\qquad\times \int_{u=0}^{t}\int_{y=0}^{(Bu^b)^{1/a}} (t-u)^{(k-1)(a+b)/a}\big(B-\tfrac{y^a}{u^b}\big)^{(k-1)/a} dy du.\end{aligned}$$

Then, by a change of variable $z=y^a/(Bu^b)$, we get that

$$\begin{aligned}\int_{y=0}^{(Bu^b)^{1/a}} \big(B-\tfrac{y^a}{u^b}\big)^{(k-1)/a} dy &= B^{(k-1)/a}\int_0^1 (1-z)^{(k-1)/a}\, \tfrac{1}{a} z^{1/a-1} B^{1/a} u^{b/a} dz \\ &= \tfrac{1}{a}\, A^{k/a} u^{b/a}\, \frac{\Gamma\big((k-1)/a+1\big)\Gamma(1/a)}{\Gamma(k/a)}\,.\end{aligned}$$

Moreover, we also have, with a change of variable $w=u/t$

$$\begin{aligned}\int_{u=0}^{t} u^{b/a}(t-u)^{(k-1)(a+b)/a} du &= t^{(k-1)(a+b)/a+b/a+1}\int_0^1 w^{b/a}(1-w)^{(k-1)(a+b)/a} dw \\ &= t^{k(a+b)/a}\frac{\Gamma\big(b/a+1\big)\Gamma\big((k-1)(a+b)/a+1\big)}{\Gamma\big(k(a+b)/a+1\big)}\,,\end{aligned}$$

and this completes the induction.

For the inequality in the second part of the lemma, we use again Stirling's formula to control $\Gamma\big(k(a+b)/a+1\big)$ and $\Gamma\big(k/a+1\big)$, and we obtain

$$\mathrm{Vol}(E_k(t,B)) \leqslant \frac{c}{k\sqrt{1/a}\sqrt{(a+b)/a}}\left(\frac{\frac{2}{a}\Gamma(\frac{1}{a})\Gamma(\frac{a+b}{a}) \times B^{1/a}t^{(a+b)/a}}{\big(e^{-1}/a\big)^{1/a}\big(e^{-1}(a+b)/a\big)^{(a+b)/a}k^{1/a}k^{(a+b)/a}}\right)^k .$$

Thanks to the asymptotics of $\Gamma(\alpha)$ as $\alpha \to +\infty$ and $\alpha \to 0$, we find that there is a constant c such that for all a, b

$$\mathrm{Vol}(E_k(t,B)) \leqslant \frac{a}{\sqrt{a+b}}\left(\frac{cB^{1/a}t^{(a+b)/a}}{(a+b)^{1/2}k^{1/a}k^{(a+b)/a}}\right)^k .$$

□

Again, as for the Hölder case, we use this Lemma to control the probability that $\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x})$ is larger than some k: similarly to (5.2)-(5.3), we get that

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x}) \geqslant k\Big) \leqslant \binom{m}{k} \times \frac{\mathrm{Vol}\big(E_k(t,B)\big)}{(2tx)^k/k!} \leqslant \left(\frac{CB^{1/a}t^{(a+b)/a}m}{txk^{(a+b+1)/a}}\right)^k ,$$

where we used Lemma 5.2 together with $\binom{m}{k} \leqslant m^k/k!$. This gives the upper bound (2.9).

Lower bound. The proof is very similar to that in the Hölder case: for any $k \geqslant 1$, consider for $1 \leqslant i \leqslant 4k$ the sub-boxes of $\Lambda_{t,x}$

$$\mathcal{B}_i := \Big[\frac{(i-1)t}{4k}, \frac{it}{4k}\Big) \times \Big[-\frac{B^{1/a}(t/4)^{b/a}}{2k^{(b+1)/a}} \wedge x, \frac{B^{1/a}(t/4)^{b/a}}{2k^{(b+1)/a}} \wedge x\Big] .$$

Then, notice that if there are at least k boxes among $\{\mathcal{B}_{2i}\}_{1\leqslant i \leqslant 2k}$ containing (at least) one point, then this set of k points has an entropy which is bounded by

$$k \times \frac{(B^{1/a}(t/4)^{b/a}k^{-(b+1)/a})^a}{(t/4k)^b} \leqslant B .$$

Hence, we get similarly to (5.5)-(5.6) that

$$\begin{aligned}\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x}) \leqslant k\Big) &\leqslant \binom{2k}{k}\mathbb{P}\Big(\Upsilon_m \cap \bigcup_{i=1}^k \mathcal{B}_i = \varnothing\Big)\\ &\leqslant 2^{2k}\Big(1-\frac{B^{1/a}t^{b/a}}{4^{b/a}k^{(a+b)/a}x} \wedge \frac{1}{4}\Big)^m . \end{aligned} \tag{5.7}$$

In the second inequality we again used that Υ_m is a set of m independent random variables uniform in $\Lambda_{t,x}$ (of volume $2tx$), and that $\bigcup_{i=1}^k \mathcal{B}_i$ has here a volume of $\frac{B^{1/a}(t/4)^{(a+b)/a}}{k^{(b+1)/a}} \wedge \frac{tx}{2}$. Therefore, we obtain that

$$\mathbb{P}\Big(\mathcal{L}_m^{(\mathcal{E}_{a,b}^B)}(\Lambda_{t,x}) \leqslant k\Big) \leqslant \exp\left\{ck\Big(1-c\Big(\frac{B^{1/a}t^{b/a}}{xk^{(a+b)/a}} \wedge 1\Big)\frac{m}{k}\Big)\right\},$$

which concludes the proof of (2.10).

5.3. **Non-directed E-LPP.** We proceed analogously to the two previous sections. The calculations are similar to the Section 5.1-5.2. Recall that we only deal with the Entropy case, since the Hölder case is identical, see (3.3)-(3.4).

Upper bound. Let us define the sets of k elements (with order) of $\mathbb{R}^2$ that have an entropy up to time horizon t smaller than B,

$$\begin{aligned}\mathrm{E}_k(t,B) &= \Big\{\Delta = (x_i)_{1\leqslant i\leqslant k}\,;\, \mathbf{Ent}_{a,b}(t,\Delta) \leqslant B\Big\}\\ &= \Big\{\Delta = (x_i)_{1\leqslant i\leqslant k}\,;\, \sum_{i=1}^{k} \|x_i - x_{i-1}\|^{a/(b+1)} \leqslant D\Big\} =: \tilde{\mathrm{E}}_k\big(D\big)\,,\end{aligned}$$

with $D = (Bt^b)^{1/(b+1)}$ – we used (3.4) to get the second equality. Here again, we are able to compute the volume of $\tilde{\mathrm{E}}_k\big(D\big)$. For simplicity, let us set $\gamma = (b+1)/a$ (as in Remark 3.1).

Lemma 5.3. *For any $D > 0$, we have for any $k \geqslant 1$*

$$\mathrm{Vol}\big(\tilde{\mathrm{E}}_k(D)\big) = (2\pi\gamma)^k \frac{\Gamma(2\gamma)^k}{\Gamma(2k\gamma+1)} D^{2k\gamma}.$$

In particular, recalling $\gamma = (b+1)/a$ and $D = (Bt^b)^{1/(b+1)}$, it gives that there exists some constant $C = C_{a,b}$ such that

$$\mathrm{Vol}(\mathrm{E}_k(t,B)) \leqslant \Big(\frac{C(Bt^b)^{2/a}}{k^{2(b+1)/a}}\Big)^k.$$

Proof. We prove the first part of Lemma 5.3 by iteration. Note that we easily have that $\tilde{\mathrm{E}}_1(D)$ is a disk of radius D^γ, so that $\mathrm{Vol}\big(\tilde{\mathrm{E}}_1(D)\big) = \pi D^{2\gamma}$. For the iteration, we use for $k \geqslant 2$ the recursion formula

$$\begin{aligned}\mathrm{Vol}\big(\tilde{\mathrm{E}}_k(D)\big) &= \int_0^{D^\gamma} 2\pi r \mathrm{Vol}\big(\tilde{\mathrm{E}}_{k-1}(D - r^{1/\gamma})\big) dr\\ &= (2\pi)^k \gamma^{k-1} \frac{\Gamma(2\gamma)^{k-1}}{\Gamma(2(k-1)\gamma+1)} \int_0^{D^\gamma} r\big(D - r^{1/\gamma}\big)^{2(k-1)\gamma} dr\,.\end{aligned}$$

Then a change of variable $u = D^{-1} r^{1/\gamma}$ gives that

$$\int_0^{D^\gamma} r\big(D - r^{1/\gamma}\big)^{2(k-1)\gamma} dr = \gamma D^{2k\gamma} \int_0^1 u^{2\gamma-1}(1-u)^{2(k-1)\gamma} du = \gamma D^{2k\gamma} \frac{\Gamma(2\gamma)\Gamma(2(k-1)\gamma+1)}{\Gamma(2k\gamma+1)}\,,$$

which concludes the induction.

For the second part of the lemma, we use again Stirling's formula to get that $\Gamma(2k\gamma+1) \geqslant (ck)^{2k\gamma}$, and we obtain

$$\mathrm{Vol}\big(\tilde{\mathrm{E}}_k(D)\big) \leqslant \Big(\frac{2\pi\gamma\Gamma(2\gamma)D^{2\gamma}}{ck^{2\gamma}}\Big)^k.$$

Recalling $D = (Bt^b)^{1/(b+1)}$ and $\gamma = (b+1)/a$, we get the conclusion. ☐

We then use this Lemma to control the probability that $\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r)$ is larger than some k: similarly to (5.2)-(5.3), we get that

$$\mathbb{P}\Big(\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r) \geqslant k\Big) \leqslant \mathbb{E}[\mathscr{N}_k] = m^k \mathbb{P}\big((Z_i)_{1\leqslant i\leqslant k} \in \mathrm{E}_k(t,B)\big).$$

Here, $\mathscr{N}_k$ is the number of k-uples in Υ_m that are $\mathscr{E}_{a,b}^{t,B}(t)$ compatible, and $(Z_i)_{1\leqslant i\leqslant k}$ are i.i.d. random variables, uniform in Λ_r the disk of radius r. Then, with Lemma 5.3, we get that

$$\mathbb{P}\Big(\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r) \geqslant k\Big) \leqslant m^k \frac{\mathrm{Vol}\big(\mathrm{E}_k(t,B)\big)}{(\pi r^2)^k} \leqslant \Big(\frac{C(Bt^b)^{2/a} m}{r^2 k^{2(b+1)/a}}\Big)^k. \tag{5.8}$$

This gives the upper bound (3.6).

Lower bound. The proof is analogous to that in the directed context, with some adaptations to deal with the non-directedness which make the proof more technical.

We consider a partition of the plan into small squares of side $\delta := \pi r/\sqrt{m}$: for any $x \in (\delta\mathbb{Z})^2$ we let $\mathcal{B}_x$ be the square of side δ centered at x. It is easy to see that there are at least $m/4$ disjoint squares $\mathcal{B}_x$ (provided that m is large enough) that can be placed into a rectangle (inscribed in Λ_r) ordered as follow: we let $x_0 = 0$ and then we enumerate $x_1, \dots, x_{m/4}$ following a spiral in a clockwise way, in order to have that any two consecutive $\mathcal{B}_{x_i}, \mathcal{B}_{x_{i+1}}$ are adjacent (see Figure 4).

Then, since a square $\mathcal{B}_x$ has volume $\pi^2 r^2/m$ (and recalling Λ_r has volume πr^2), $\mathcal{B}_x$ contains at least one point of Υ_m with probability $1-(1-\pi/m)^m \geqslant 1-e^{-\pi}$. We define $Q_{m/4}$ the number of non-empty squares among $\mathcal{B}_{x_0}, \dots, \mathcal{B}_{x_{m/4}}$, and we define iteratively the indices I_j of the non-empty squares, by $I_0 = 0$ and for $1 \leqslant j \leqslant Q_{m/4}$

$$I_j = \inf\left\{i > I_{j-1} \ ; \ \mathcal{B}_{x_i} \cap \Upsilon_m \neq \varnothing\right\}.$$

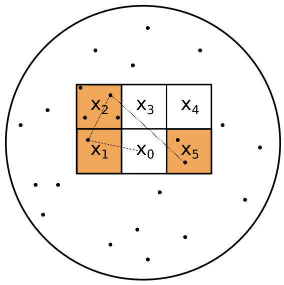


FIGURE 4. In the picture we put $m = 24$ points uniformly on Λ_r and we consider a rectangle built by $6 = m/4$ squares $\mathcal{B}_{x_0}, \cdots, \mathcal{B}_{x_5}$ enumerated following a spiral in a clockwise way starting from the origin. Then we consider the non-empty rectangles (in orange) and their indices. In this example we have $I_1 = 1, I_2 = 2, I_3 = 5$. Finally we draw a path starting from the origin and collecting one point in exactly all $\mathcal{B}_{I_1}, \cdots, \mathcal{B}_{I_3}$.

For $k \geqslant 1$, and if $Q_{m/4} \geqslant k$, we may consider a path Δ collecting one point in exactly all $\mathcal{B}_{x_{I_1}}, \dots, \mathcal{B}_{x_{I_k}}$: the entropy of such Δ is bounded by (see Figure 4)

$$\frac{1}{t^b}\Big(\sum_{j=1}^{k}\big(4(I_j - I_{j-1})\delta\big)^{a/(b+1)}\Big)^{b+1} \leqslant \frac{4^a r^a}{t^b m^{a/2}}\Big(\sum_{j=1}^{k} U_j\Big)^{b+1},$$

where we set $U_j := (I_j - I_{j-1})^{a/(b+1)}$. Therefore, for $\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r)$ to be smaller or equal than k, one needs to have either $Q_{m/4} < k$ or that the entropy of Δ chosen above is larger than B: this leads to

$$\mathbb{P}\Big(\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r) \leqslant k\Big) \leqslant \mathbb{P}\big(Q_{m/4} < k\big) + \mathbb{P}\Big(Q_{m/4} \geqslant k\,,\ \sum_{j=1}^{k} U_j > \Big(\frac{Bt^b m^{a/2}}{4^a r^a}\Big)^{1/(b+1)}\Big). \tag{5.9}$$

For the first term, and for $k \leqslant \varepsilon^2 m/4$ (with $\varepsilon > 0$ small, fixed in a moment), we realize that $Q_{m/4} < k$ implies that there are at least $(1-\varepsilon^2)m/4$ empty squares, which gives by a union bound that

$$\mathbb{P}\big(Q_{m/4} < k\big) \leqslant \binom{m/4}{(1-\varepsilon^2)m/4}\mathbb{P}\Big(\Upsilon_m \cap \bigcup_{i=1}^{(1-\varepsilon^2)m/4} \mathcal{B}_{x_i} = \varnothing\Big) \leqslant e^{c_\varepsilon m}\Big(1 - \frac{(1-\varepsilon^2)\pi}{4}\Big)^m$$

where for the second inequality, we used that the volume of $\bigcup_{i=1}^{(1-\varepsilon^2)m/4} \mathcal{B}_{x_i}$ is $(1-\varepsilon^2)\pi^2 r^2/4$. We note that the constant c_ε goes to 0 as ε goes to 0: we can therefore fix $\varepsilon > 0$ sufficiently small so that

$$\mathbb{P}\big(Q_{m/4} < k\big) \leqslant e^{-\pi m/8} \quad \text{for all} \quad k \leqslant \varepsilon^2 m/4. \tag{5.10}$$

For the second term in (5.9), let us write $V := k^{-1}\big(Bt^b m^{a/2}/r^a\big)^{1/(b+1)}$ – we will consider only the case when V is large –, so that we need to bound

$$\mathbb{P}\Big(Q_{m/4} \leqslant k\,,\ \sum_{j=1}^k U_j > kV\Big) \leqslant \mathbb{P}\big(N_k > \varepsilon m\big) + \mathbb{P}\Big(Q_{m/4} \leqslant k\,, N_k \leqslant \varepsilon m\,,\ \sum_{j=1}^k U_j > kV\Big)\,, \tag{5.11}$$

where N_k denotes the total number of points in the non-empty squares $\mathcal{B}_{x_{I_1}}, \dots, \mathcal{B}_{x_{I_k}}$. We easily have that

$$\mathbb{P}\big(N_k > \varepsilon m\big) \leqslant \frac{1}{e^{-ck}} \binom{m}{\varepsilon m} \Big(\frac{\pi k}{m}\Big)^{\varepsilon m} \leqslant e^{ck} \frac{(\pi k)^{\varepsilon m}}{(\varepsilon m)!}\,,$$

where the denominator in the first inequality comes from the fact that we work conditionally on the fact that k squares are non-empty (which has probability bounded below by e^{-ck}). Hence, since we work with $k \leqslant \varepsilon^2 m/4$, and provided that ε has been fixed small enough, we get that there is a constant $c > 0$ such that $\mathbb{P}(N_k > \varepsilon m) \leqslant e^{-cm}$.

For the last part, note that since the squares $\mathcal{B}_x$ are exchangeable, we can control for $1 \leqslant i_1 < \cdots < i_k \leqslant m/4$

$$\begin{aligned}
&\mathbb{P}\big(I_1 = i_1, \dots, I_k = i_k; N_k \leqslant \varepsilon m\big) \\
&\qquad = \sum_{\substack{n_1,\dots,n_k \\ 1 \leqslant n_1+\cdots+n_k \leqslant \varepsilon m}} \binom{m}{n_1,\dots,n_k} \Big(\frac{\pi}{m}\Big)^{n_1+\cdots+n_k} \Big(1 - \frac{\pi i_k}{m}\Big)^{m-(n_1+\cdots+n_k)} \\
&\qquad \leqslant \Big(1 - \frac{\pi i_k}{m}\Big)^{(1-\varepsilon)m} \sum_{\substack{n_1,\dots,n_k \\ 1 \leqslant n_1+\cdots+n_k \leqslant \varepsilon m}} \frac{\pi^{n_1}}{n_1!} \cdots \frac{\pi^{n_k}}{n_k!} \leqslant e^{-(1-\varepsilon)\pi i_k} e^{\pi k}\,.
\end{aligned}$$

Where we used that in order to have $I_1 = i_1, \dots, I_k = i_k$ there must be exactly k non-empty squares among the first i_k (with $n_1, \dots, n_k$ points in them) and $i_k - k$ empty. The remaining $m - (n_1 + \cdots + n_k)$ points must be outside the first i_k squares. For the second inequality, we used that $n_1 + \cdots + n_k \leqslant \varepsilon m$, and that the multinomial coefficient is bounded by $m^{n_1+\cdots+n_k}/(n_1! \cdots n_k!)$. Hence, there is a constant c such that

$$\mathbb{P}\big(I_1 = i_1, \dots, I_k = i_k; N_k \leqslant \varepsilon m\big) \leqslant e^{ck} \times \mathbb{P}\big(G_j = i_j - i_{j-1} \ \text{ for all } 1 \leqslant j \leqslant k\big)\,,$$

where $(G_j)_{j\geqslant 1}$ are i.i.d. geometric random variables, of parameter $1 - e^{-(1-\varepsilon)\pi}$. We therefore obtain that, provided that V is large enough

$$\begin{aligned}
\mathbb{P}\Big(Q_{m/4} \leqslant k\,, N_k \leqslant \varepsilon m\,,\ \sum_{j=1}^k U_j > kV\Big) &\leqslant e^{ck}\,\mathbb{P}\Big(\sum_{j=1}^k (G_j)^{a/(b+1)} > kV\Big) \\
&\leqslant e^{-c'kV}\,.
\end{aligned} \tag{5.12}$$

To conclude, we have obtained that there are constants such that for $k \leqslant \varepsilon^2 m/4$, and for $V := k^{-1}\big(Bt^b m^{a/2}/r^a\big)^{1/(b+1)}$ large enough,

$$\mathbb{P}\Big(\mathscr{L}_m^{(\mathscr{E}_{a,b}^{t,B})}(\Lambda_r) \leqslant k\Big) \leqslant e^{-cm} + e^{-c'kV}\,. \tag{5.13}$$

One obtains (3.7) by observing that when V is small $e^{-ck(V-1)}$ is larger than 1. The statements holds for all $k \leqslant m$ by adjusting the constants.

Appendix A. Further simulations and conjectures

In this appendix, we present further simulations, that help us make some predictions on the values of the constants in (2.13), and support the belief that the model is in the KPZ universality class. We treat only the directed case because in the non-directed case simulations are much more greedy and do not bring any convincing insight – we admit that our algorithm could be improved, but our goal is simply to hint for some conjectures, and our simulations fills that role perfectly. We start by commenting simulations in the H_γ-LPP case, where simulations are exact (and efficient), before we turn to the E-LPP case.

A.1. **Directed H_γ-LPP.** For the H_γ-LPP, we performed two different simulations, in the Poissonian context of Section 2.3 —focusing on the point-to-point H_γ-LPP, so we write $\mathcal{L}(t,0)$ for $\mathcal{L}_\lambda^{(\mathcal{H}_\gamma^A)}(t,0)$.

(1) We ran (a few) simulations for $t = 1000$ (with $\lambda, A = 1$, restricting to the box $[0,t] \times [-t^{2/3}, t^{2/3}]$), in order to test the value of the constant $\mathtt{C} = \mathtt{C}_{1,1}(0) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}(t,0)$ in (2.17). The results are presented in Figure 6, and commented below.
(2) In order to test the convergence in distribution of the recentered H_γ-LPP, $t^{-1/3}(\mathcal{L}(t,0) - \mathtt{C}t)$, we built histograms by running $k = 10^3$ simulations of the H_γ-LPP for $t = 500$ (with $\lambda, A = 1$, restricting to the box $[0,t] \times [-t^{2/3}, t^{2/3}]$), for three values $\gamma = 0$, $\gamma = 0.5$, $\gamma = 1.5$. The results are collected in Figure 7, and commented below.

(1) *Value of the constant.* Let us present here our results for simulations for the value of the constant, performed for $t = 1000$, in the box $[0,t] \times [-t^{2/3}, t^{2/3}]$, with intensity $\lambda = 1$ and with a constraint $A = 1$.

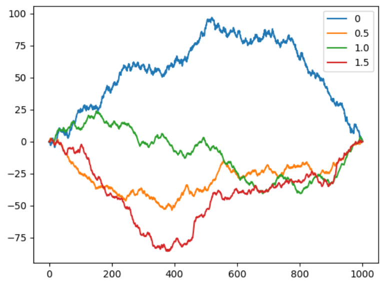


Figure 5. Simulations of optimal paths for the H_γ-LPP with $t = 1000$ (intensity $\lambda = 1$, constraint $A = 1$), for different values of γ. The same set of points is used in all four simulations. For $\gamma = 0$ we have here $\mathcal{L}(t,0) = 2707$, for $\gamma = 0.5$ $\mathcal{L}(t,0) = 1715$, for $\gamma = 1$ $\mathcal{L}(t,0) = 1408$, and for $\gamma = 1.5$ $\mathcal{L}(t,0) = 1238$. We refer to Figure 6 for a graph presenting how the constant $\mathtt{C}_{1,1} = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}(t,0)$ depends on γ.

Our simulations are in accordance with the fact that $\frac{1}{t}\mathcal{L}(t,0)$ converges a.s. to some constant, whose dependence on γ is presented in Figure 6 (we present the result of only one simulation, but several simulations give values for $\frac{1}{t}\mathcal{L}(t,0)$ very close to those presented here). In view of the dependence on γ of the constant c_1 in Theorem 2.1 (see in particular (5.1)), a wild guess is that the constant is proportional to $(1+\gamma)^{-1}\Gamma(1+\gamma)^{1/(1+\gamma)}$: the dotted grey line in Figure 6 represents the function $\gamma \mapsto \frac{2^{3/2}}{1+\gamma}\Gamma(1+\gamma)^{1/(1+\gamma)}$ – the factor $2^{3/2}$ is chosen so that it fits the value $\sqrt{2}$ when $\gamma = 1$, corresponding to the standard Lipschitz

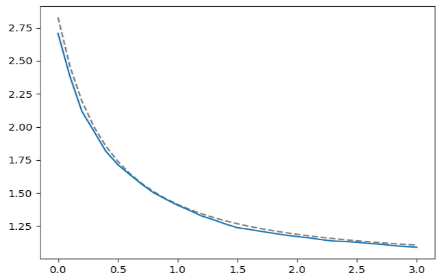
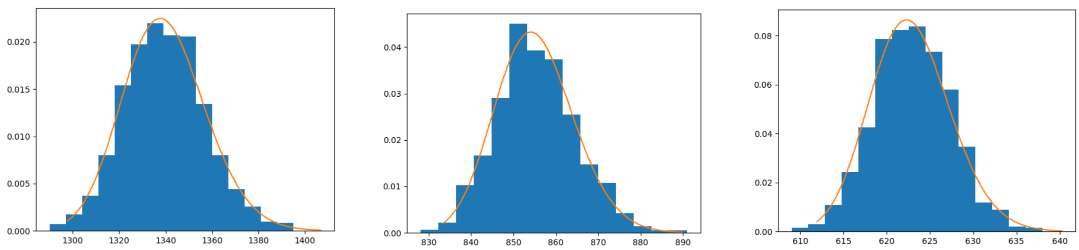


FIGURE 6. Approximated values of the H_γ-LPP constant: the function represents the value of $\frac{1}{t}\mathcal{L}(t,0)$ with $t = 1000$ (intensity $\lambda = 1$, constraint $A = 1$), for different values of $\gamma \in [0,3]$. The dotted grey line represents the function $\gamma \mapsto \frac{2^{3/2}}{1+\gamma}\Gamma(1+\gamma)^{1/(1+\gamma)}$, which seems to be a good candidate to fit the values of $\frac{1}{t}\mathcal{L}(t,0)$. We refer to Figure 5 for the corresponding paths for $\gamma = 0, 0.5, 1, 1.5$.

LPP (the missing factor $\sqrt{2}$ comes from the length of the diagonal in Hammersley's LPP process). The two curves match quite closely, but they seem to disagree when $\gamma = 0$ (the constant $\mathtt{C}_{1,1}$ seems very close to 2.75, whereas $2^{3/2} \approx 2.83$).

(2) *Convergence of the recentered and renormalized LPP.* In order to test the convergence in distribution of $t^{-1/3}(\mathcal{L}(t,0) - \mathtt{C}_{1,1}t)$, we performed 1000 simulations for the point-to-point H_γ-LPP with $t = 500$ (again with intensity $\lambda = 1$, and constraint $A = 1$, in the box $[0,t] \times [-t^{2/3}, t^{2/3}]$), for the three values $\gamma = 0$, $\gamma = 0.5$ and $\gamma = 1.5$. The histograms presented in Figure 7 seem to confirm the convergence in distribution to a Tracy-Widom GUE limit.



(a) $\gamma = 0$. (b) $\gamma = 0.5$. (c) $\gamma = 2$.

FIGURE 7. Histograms of $k = 10^3$ simulations of the point-to-point H_γ-LPP in Poisson environment (with $\lambda = 1$ and $A = 1$) with $t = 500$. The three subfigures (a), (b) and (c) correspond to the cases $\gamma = 0$, $\gamma = 1/2$ and $\gamma = 3/2$ respectively. In each case, we also present the graph of the Tracy-Widom GUE density, after a recentering by $\mathtt{C}_\gamma t$ (with $\mathtt{C}_\gamma \approx 2.75, 1.75, 1.26$ from left to right), and a renormalization by $c_\gamma t^{1/3}$ (with $c_\gamma \approx 2.5, 1.3, 0.65$ from left to right).

All together, this leads to a (far-reaching) conjecture, for the (point-to-point) H_γ-LPP.

Conjecture A.1. *For every $\gamma \geqslant 0$, there exists a constant $\mathtt{C}_\gamma$ (equal to $\frac{2^{3/2}}{1+\gamma}\Gamma(1+\gamma)^{1/(1+\gamma)}$?) and a constant c_γ such that, for the point-to-point H_γ-LPP in Poisson environment with intensity $\lambda = 1$ and γ-Hölder constraint $A = 1$, we have*

$$\frac{\mathcal{L}(t,0) - \mathtt{C}_\gamma\, t}{c_\gamma\, t^{1/3}} \xrightarrow[t\to\infty]{(d)} F_{GUE} \quad \text{as } t \to +\infty\,. \tag{A.1}$$

A.2. **Directed E-LPP.** As far as the directed E-LPP is concerned, we also performed simulations in the setting of Section 2.3 with $t = 100$, with a Poisson intensity $\lambda = 1$ and a constraint $B = 1$ (within the box $[0,t] \times [-t^{2/3}, t^{2/3}]$). Simulations are much less efficient, and the simulated annealing procedure only gives an approximate (under-estimated) value for $\mathcal{L}(t,0) = \mathcal{L}_\lambda^{(\mathcal{E}^B_{a,b})}(t,0)$.

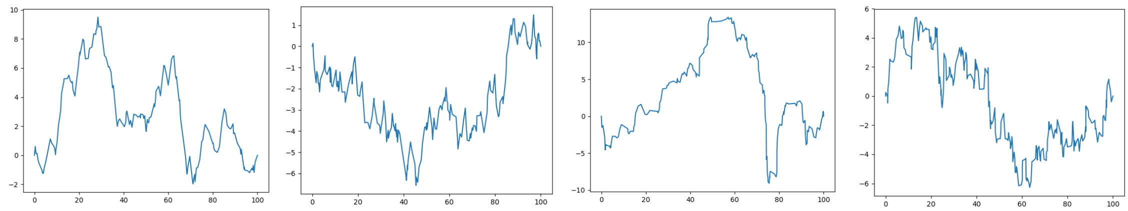


FIGURE 8. Simulation of Poisson point-to-point E-LPP with $t = 100$ (with intensity $\lambda = 1$ and constraint $B = 1$), via a simulated annealing procedure. The plots represents a path which collects a number of points that approximate $\mathcal{L}(t,0)$, with different parameters a, b, in order to test the value of the constant $\mathtt{C}_{1,1}(0) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}(t,0)$ in (2.13). From left to right we have: $a = 2, b = 1$ ($\mathtt{C} \approx 1.83$), $a = 4, b = 1$ ($\mathtt{C} \approx 1.96$), $a = 1, b = 0$ ($\mathtt{C} \approx 2.08$), $a = 2, b = 0$ ($\mathtt{C} \approx 2.55$).

Figure 8 presents some simulations to test the dependence of the constant $\mathtt{C}_{1,1}(0) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}(t,0)$ on the parameters a, b. We give some values for the constant, and the only conjecture we may risk to formulate (thanks to simulations for others values of a, b that we do not present here) is that the constant should be non-decreasing in a and non-increasing in b. Further conclusions are hard to draw from these simulations.

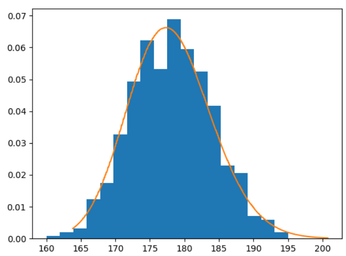


FIGURE 9. Histogram of 1000 realizations of $\mathcal{L}(t,0)$ for $t = 100$ (with intensity $\lambda = 1$ and constraint $B = 1$), with $a = 2, b = 1$. We also plotted the graph of the GUE density, centered by $\mathtt{C}_{a,b} t$ with $\mathtt{C}_{a,b} \approx 1.89$, and rescaled by $c_{a,b}\, t^{1/3}$ with $c_{a,b} \approx 1.4$.

The histogram presented in Figure 9 makes it natural to conjecture that for every a, b there exists some constant $\mathtt{C}_{a,b}$, such that for the point-to-point E-LPP in Poisson environment with intensity $\lambda = 1$ and entropy constraint $B = 1$, we have the convergence in distribution

$$\frac{\mathcal{L}(t,0) - \mathtt{C}_{a,b}\, t}{c_{a,b}\, t^{1/3}} \xrightarrow{(d)} F_{GUE} \quad \text{as } m \to \infty\,,$$

with $c_{a,b}$ a renormalization constant.

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