Phénomènes de localisation et d'universalité pour des polymères aléatoires

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1. Polymers

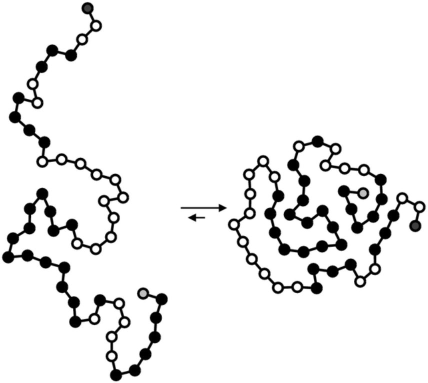
2. Pinning Model

3. Universality of the Pinning Model

4. Pinning Model with Heavy Tailed disorder

5. Perspectives

Polymers



Interactions with

- Itself
- External environment

Depend on some parameters

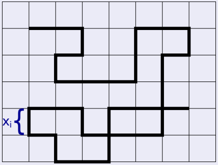
Aim

- Spatial configuration
- Phase transition? Critical Points?

Intermezzo: Some Basic Probabilistic Processes

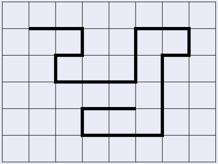
Random Walk on $\mathbb{Z}^d$

$$S_n = \sum_{i=1}^n X_i, \qquad X_i = \text{increments}.$$



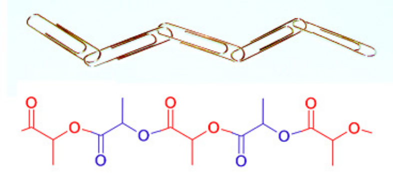
Self Avoiding Random Walk (SAW) on $\mathbb{Z}^d$

Conditioned to visit at most ones each state.

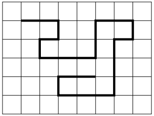


Abstract Monomers and Abstract Polymers

Increment $X_i \Leftrightarrow$ a monomer



Use SAW:



Abstract Polymer (N monomers):
N-increments of a SAW

Abstract Monomers and Abstract Polymers

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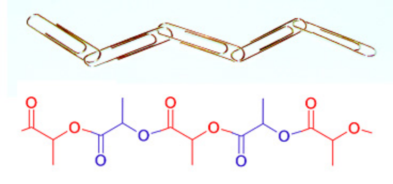

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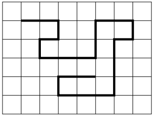
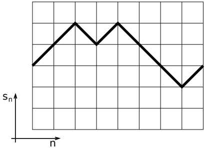


Abstract Polymer (*N* monomers):
N-increments of a SAW

SAW: challenging object!

Subclass: Directed Random Walks (Directed Polymers)



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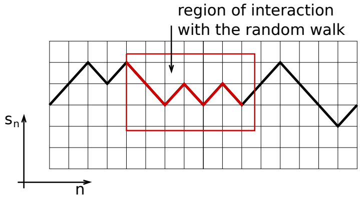
Pinning Model

Pinning Model: interactions polymer ⇔ environment.



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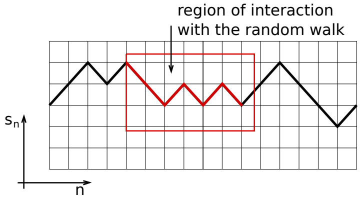
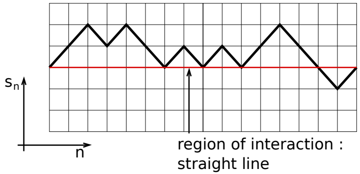
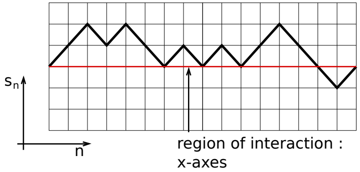


Interaction with a membrane ⇔ region = straight line



Henceforth S one-dimensional.

Pinning Model



Modify Random Walk S:
reward/penalty whenever S visits 0

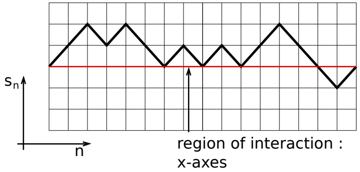
$$P^{\omega}_{N,\beta,h}(s_1,\dots,s_N) = \frac{1}{Z^{\omega}_{N,\beta,h}} \exp\left\{\sum_{i=1}^{N} g_i^{\omega}(\beta,h)\,\mathbb{1}_{\{s_i=0\}}\right\} \cdot P(s_1,\dots,s_N).$$

N : polymer length

$$g_i^{\omega}(\beta,h) = \begin{cases} h, & \text{Homogeneous model,} \\ \beta\omega_i + h, & \text{Disordered model.} \end{cases} \quad \omega = (\omega_i)_{i\in\mathbb{N}} \text{ disorder}$$

$Z^{\omega}_{N,\beta,h}$ Partition Function

Pinning Model



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$\omega = (\omega_i)_{i \in \mathbb{N}}$ disorder

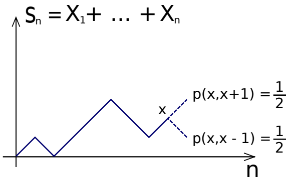
$Z^{\omega}_{N,\beta,h}$ Partition Function

Remark:

Need only $\{n : S_n = 0\}$ to define the model.

Random Walk choice

symmetric simple random walk



$$P(S_n = 0 \text{ for the first time}) \underset{n\to\infty}{\sim} \frac{c}{n^{\frac{3}{2}}}$$

Random Walk choice

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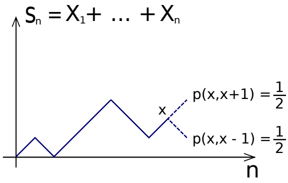

$$P(S_n = 0 \text{ for the first time}) \underset{n\to\infty}{\sim} \frac{c}{n^{\frac{3}{2}}}$$

Generalize

$$P(S_n = 0 \text{ for the first time}) \underset{n\to\infty}{\sim} \frac{c_S}{n^{1+\alpha}}, \qquad \alpha > 0.$$

Explicit examples: $\alpha \in (0, 1)$ Bessel-like Random walk.

▼ ■

Disorder Assumptions

Disorder ω: *quenched* realization of an *i.i.d.* random sequence.

$$\mathbb{E}(\omega_1) = 0, \quad \mathbb{V}ar(\omega_1) = 1, \quad \Lambda(\beta) = \log \mathbb{E}(e^{\beta\omega_1}) < \infty, \forall \beta \text{ small.}$$

Examples:

- ω_1 bounded,
- ω_1 Gaussian.

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Examples:

- ω_1 bounded,
- ω_1 Gaussian.

Different interesting choice:

heavy-tail case $\mathbb{P}(\omega_1 > t) \underset{t\to\infty}{\sim} c\, t^{-\xi}$

Our Goal

Goal: behavior of S under $P^{\omega}_{\beta,h,N}$ when N gets large.

N	"abstract polymer" length,
h	Homogeneous parameter,
β	Tunes the presence of the disorder ω

- localized? de-localized?
- phase transition on β, h? Critical point?

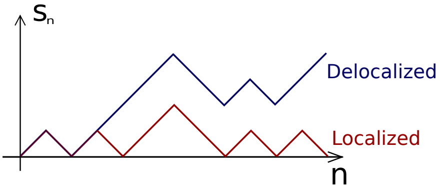
Localization/Delocalization

Free Energy

$$\boxed{F^{(\alpha)}(\beta,h) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log Z^{\omega}_{\beta,h,N}\right]}$$

$$\frac{\partial}{\partial h}F^{(\alpha)}(\beta,h) = \lim_{N\to\infty} \mathbb{E}E^{\omega}_{\beta,h,N}\left(\frac{\sharp\{n \le N \,:\, S_n = 0\}}{N}\right) \qquad (*)$$

$\forall\,\beta \ge 0$ there exists a critical point $h_c(\beta)$



- $h > h_c(\beta)$ localization, $(*) > 0$,
- $h < h_c(\beta)$ de-localization, $(*) = 0$.

Analysis of the model

Goal: understand $h_c(\beta)$.

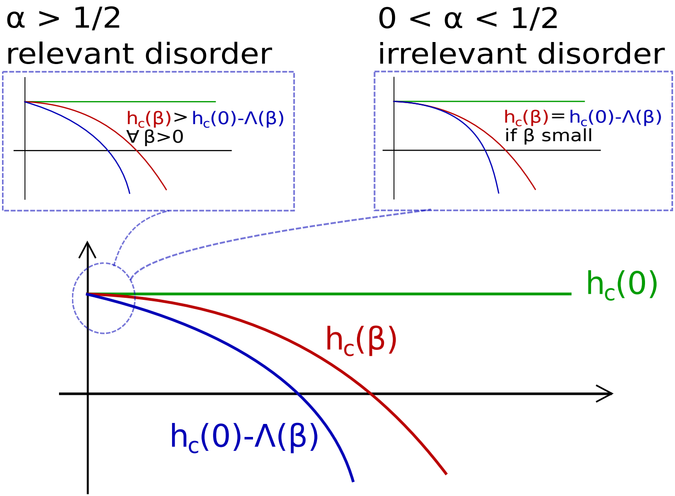
- Homogeneous model $h_c(0)$ explicit ($= 0$ if S is recurrent).

It provides Lower/Upper bounds

$$\boxed{h_c(0) - \Lambda(\beta) \le h_c(\beta) < h_c(0),}$$

- $h_c(0) - \Lambda(\beta)$ *annealed critical point*.

Relevance/Irrelevance of the disorder



Aim: When $\alpha > 1/2$, asymptotics of $h_c(\beta)$ as $\beta \to 0$.

An overview of the literature

case $\alpha > 1$

Theorem (Q. Berger, F. Caravenna, J. Poisat, R. Sun, N. Zygouras, 2014)

Let $\alpha > 1$, then

$$\boxed{h_c(\beta) \underset{\beta\to 0}{\sim} \tilde{c}\beta^2,}$$

where $\tilde{c}$ is explicit depending on α and c_S.

case $\alpha \in (1/2, 1)$: several authors K. S. Alexander, B. Derrida, G. Giacomin, H. Lacoin, F. L. Toninelli and N. Zygouras (2008 – 2011).

Theorem

Let $\alpha \in (1/2, 1)$, then there exist $0 < c < C < \infty$ such that

$$\boxed{c\,\beta^{\frac{2\alpha}{2\alpha-1}} \leq h_c(\beta) \leq C\,\beta^{\frac{2\alpha}{2\alpha-1}},}$$

for β small.

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Results

(Add some Technical assumptions on Disorder and Random Walk)

Theorem (Caravenna, Toninelli, T., 2015)

Universal feature of the Free Energy

$$\boxed{\mathscr{F}^{(\alpha)}(\hat\beta, \hat h) = \lim_{\varepsilon \downarrow 0} \frac{F^{(\alpha)}(\hat\beta\, \varepsilon^{\alpha - \frac{1}{2}},\ \hat h\, \varepsilon^{\alpha})}{\varepsilon}}$$

$$\mathscr{F}^{(\alpha)}(\hat\beta, \hat h) = \beta^{\frac{2}{2\alpha-1}} \mathscr{F}^{(\alpha)}\left(1, \hat h \beta^{-\frac{2\alpha}{2\alpha-1}}\right) \quad \Rightarrow \quad \boxed{\mathscr{H}^{(\alpha)}(\hat\beta) = \hat c \hat\beta^{\frac{2\alpha}{2\alpha-1}}}$$

$\hat c$ non-trivial constant depending on α and c_S.

Theorem (Caravenna, Toninelli, T., 2015)

Universal Critical Behavior

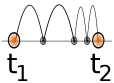
$$\boxed{h_c(\beta) \underset{\beta \to 0}{\sim} \hat c\, \beta^{\frac{2\alpha}{2\alpha-1}},}$$

Proof - Continuum Model

- $\tau := \{n \,:\, S_n = 0\} \;\Rightarrow\; \varepsilon\tau \underset{\varepsilon\downarrow 0}{\to} \hat{\tau}$
- Thm (C.S.Z.,15)

  (cond.) Pinning Model converges:

  $\beta = \hat{\beta}\varepsilon^{\alpha-\frac{1}{2}}, h = \hat{h}\varepsilon^{\alpha}$:



$$\boxed{\varepsilon\tau_{\beta,h} \xrightarrow[\varepsilon\downarrow 0]{(d)} \hat{\tau}_{\hat{\beta},\hat{h}}}$$

Continuum ingredients:

- regenerative set ($\hat{\tau}$)
- White Noise (Cont. disorder)

Problem: No Gibbs representation

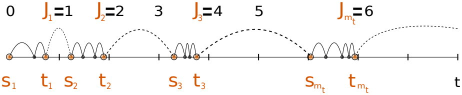
Wiener Chaos Expansion

Proof - Strategy: Coarse-Graining

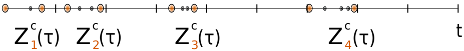
$N = \frac{t}{\varepsilon}$. Consider $\frac{1}{N}\mathbb{E}\log Z_N$

Compare $\lim_t \lim_\varepsilon$ & $\lim_\varepsilon \lim_t$

Coarse-Graining of $\tau = \varepsilon\tau$ or $\hat{\tau}$



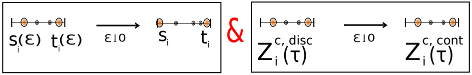
Partition function decomposition



$$Z_i^{c,\text{disc.}}(\tau/\varepsilon) = Z^c_{t/\varepsilon}\big(s_i(\varepsilon), t_i(\varepsilon)\big)$$

$$Z_i^{c,\text{cont.}}(\hat{\tau}) = \mathscr{Z}^c_t\big(s_i, t_i\big)$$

Convergence on each block



Technical difficulty

Couple together convergence of $\big(s_i(\varepsilon), t_i(\varepsilon)\big)$ with $Z^c_{t/\varepsilon}\big(a, b\big)$

$\Rightarrow$ Get: $\forall\, \eta > 0\, \exists\, \varepsilon_0\ :\ \forall\, \varepsilon < \varepsilon_0$

$$\mathscr{F}^{(\alpha)}(\hat{\beta}, \hat{h} - \eta) \leq$$

$$\varepsilon^{-1}\, F^{(\alpha)}(\hat{\beta}\, \varepsilon^{\alpha - \frac{1}{2}},\ \hat{h}\, \varepsilon^{\alpha})$$

$$\leq \mathscr{F}^{(\alpha)}(\hat{\beta}, \hat{h} + \eta)$$

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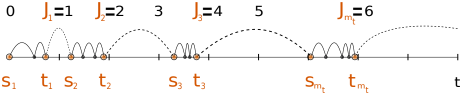

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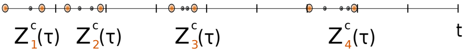

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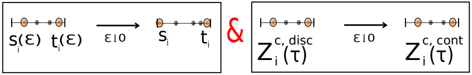

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$$\varepsilon^{-1}\, F^{(\alpha)}(\hat{\beta}\, \varepsilon^{\alpha - \frac{1}{2}},\, \hat{h}\, \varepsilon^{\alpha})$$

$$\leq \mathscr{F}^{(\alpha)}(\hat{\beta}, \hat{h} + \eta)$$

Similar to Copolymer Model

(den Hollander & Bolthausen, 1997 and Caravenna & Giacomin, 2010)

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Disorder with Heavy Tail

- Disorder ω quenched realization of an i.i.d. random sequence s.t.

$$\mathbb{P}(\omega_1 > t) \sim c t^{-\xi}, \quad t \to \infty. \quad \xi \in (0,1)$$

and ω_1 positive with a continuous distribution.

Disorder with Heavy Tail

- Disorder ω quenched realization of an i.i.d. random sequence s.t.

$$\mathbb{P}(\omega_1 > t) \sim ct^{-\xi}, \quad t \to \infty. \quad \xi \in (0,1)$$

and ω_1 positive with a continuous distribution.

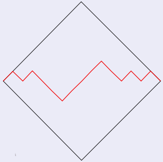
Inspired by

A. Auffinger and O. Louidor

Directed polymers in random environment with heavy tails

B. Hambly and J. B. Martin

Heavy tails in last-passage percolation



$\xi \in (0,2)$.

Renewal process

Pinning model:

$$P^{\omega}_{N,\beta,h}(s_1, ..., s_N) = \frac{1}{Z^{\omega}_{N,\beta,h}} \exp\left\{ \sum_{i=1}^{N} (\beta\omega_i + h) \mathbb{1}_{\{s_i=0\}} \right\} \mathbb{1}_{\{s_N=0\}} \cdot P(s_1, ..., s_N).$$

$$\mathbb{1}_{\{s_i=0\}} \Leftrightarrow \mathbb{1}_{\{i\in\tau\}}, \text{ where}$$

$$\tau = \{\tau_0 = 0, \tau_1, \tau_2, \cdots\} = \{n : S_n = 0\}$$ Renewal Process

Assumptions (Renewal Process)

$$\tau = \{\tau_0 = 0, \tau_1, \tau_2, \cdots\}$$ Renewal Process

Assumptions

$$K(n) := \mathrm{P}(\tau_1 = n) \text{ probability on } \mathbb{N},$$

$$\boxed{K(n) \cong e^{-Cn^{\gamma}}} \quad \boxed{\gamma \in (0,1)}$$

+ "local regularity".

Goal

Behavior of the Pinning Model ⇒

Behavior of the Renewal Process ⇒ consider $\tau/N \cap [0,1]$.

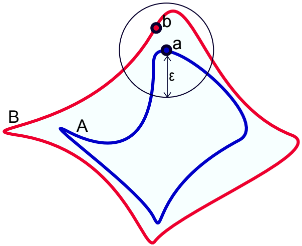
- look at $\tau/N \cap [0,1]$ as r.v. in the space of *all closed subsets of* $[0,1]$ *(which contain* $0,1$*)* .
- Equip it with Hausdorff distance:

Hausdorff distance:

$d_H(A,B) < \varepsilon \overset{\text{Def.}}{\Longleftrightarrow}$

$\forall a \in A, \exists b \in B : |a-b| < \varepsilon$

and vice-versa, interchanging A and B.



Concentration & Convergence

Take

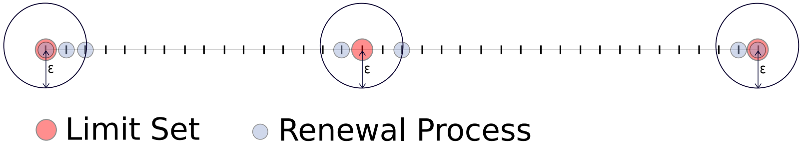
- $\beta = \beta_N = \hat{\beta} N^{\gamma - \frac{1}{\xi}} \to 0$ as $N \to \infty$.
- $h < 0$ fixed

Theorem (T., 2014)

For any $\hat{\beta} > 0$ there exists a universal *closed subset $\hat{I}^{w}_{\hat{\beta},\infty} \subset [0,1]$ such that*

$$\tau / N \cap [0,1] \stackrel{(d)}{\to} \hat{I}^{w}_{\hat{\beta},\infty}$$

in the Hausdorff metric.

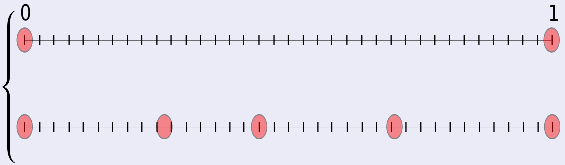


Random Critical Point

Theorem (T., 2014)

1. *There exists* $\hat{\beta}_c^{w}(h)$ *random variable s.t.*

$\hat{I}^{w}_{\hat{\beta},\infty} =$



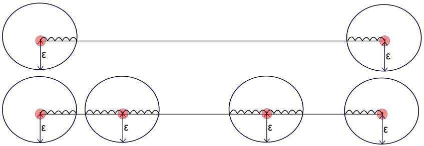
$\hat{\beta} < \hat{\beta}_c^{w}(h)$

$\hat{\beta} > \hat{\beta}_c^{w}(h)$

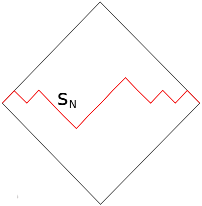
2. $\hat{\beta}_c^{w}(h) > 0$ *for a.e.-w and for any choice of* $\xi, \gamma \in (0,1)$.

$\hat{\beta}$ small, "weak disorder"

$\hat{\beta}$ big, "strong disorder"



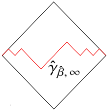
Directed polymers in random environment with heavy tails



$\xi \in (0, 2)$

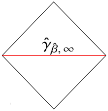
$$P_{N,\beta}(s) = \frac{e^{\beta \sum_i \omega_{i,s_i}}}{Z_{N,\beta}} P(s)$$

- $\beta = \hat{\beta} N^{1-\frac{2}{\xi}} \Rightarrow \exists\, \hat{\gamma}_{\hat{\beta},\infty} : \quad s_N \overset{(d)}{\rightarrow} \hat{\gamma}_{\hat{\beta},\infty}$



$\hat{\beta} > \hat{\beta}_c^w$

- $\exists\, \hat{\beta}_c^w$ random critical point



$\hat{\beta} < \hat{\beta}_c^w$

- $\beta_c^w = 0$ if $\xi \in [1/2, 2)$ and $\beta_c^w > 0$ if $\xi \in [0, 1/3)$

  Improved: $\beta_c^w > 0$ also $\xi \in [1/3, 1/2)$.

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Perspectives

- Universality for weak disorder : *free energy of directed polymer in random environment*.
- Structure of $\hat{I}_{\hat{\beta},\infty}$. Finite number of points?
- Renewal process with polynomial tails (and Heavy-Tailed disorder). Different limit structure?

Thanks for your attention!