Universality for the pinning model in the weak coupling regime

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— joint work with F. Caravenna and F. Toninelli —

Summary

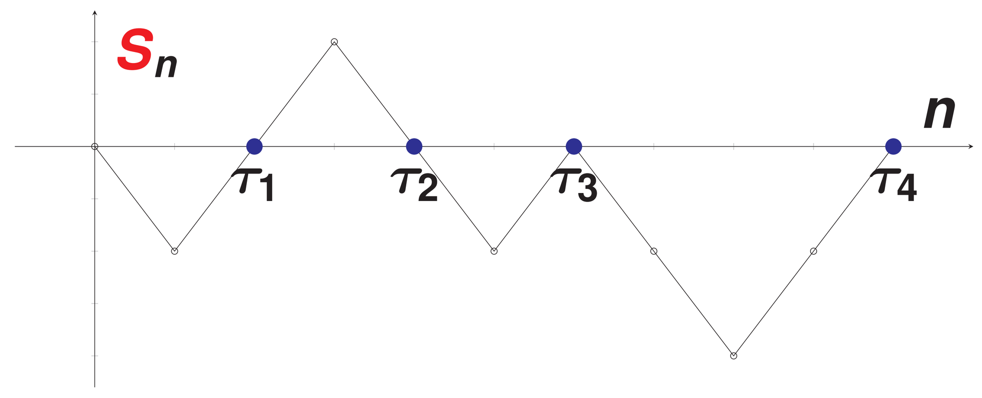
We study pinning models, i.e. we consider discrete Markov chains and perturb their behavior through a privileged interaction with a distinguished state. The return time to this state has a polynomial decay with tail exponent $\alpha \in (0,1)$. The interaction depends on an external and independent source of randomness, called disorder, which can attract or repel the Markov chain path through rewards/penalties each time the Markov chain touches the state. This can produce concentration and localization phenomena of the Markov chain path around the distinguished state, leading to a phase transition.

The main goal is to investigate this phase transition. We prove that the continuum limit introduced by [CSZ] captures the leading behavior for weak disorder of the pinning model, that is the free energy and critical curve of discrete pinning models, suitably rescaled, converge to the analogous quantities of related continuum models. This is obtained by a subtle *coarse-graining procedure*, which generalizes and refines [BH] and [CG].

The Model

Two independent random ingredients: $\mathbb{N}_0$-valued Markov chain $(S = (S_n)_{n\in\mathbb{N}}, \mathrm{P})$ and *disorder* $(\omega = (\omega_n)_{n\in\mathbb{N}}, \mathbb{P})$.

In the pinning model when the Markov chain touches the axis at time τ_i, it receives a reward/penalty given by $\beta\omega_i - \Lambda(\beta) + h$.



Precisely the pinning model is a random probability measure

$$\mathrm{P}^{\omega}_{\beta,h,N}(s_1,\cdots,s_N) = \frac{e^{\mathrm{H}^{\omega}_{\beta,h,N}(s_1,\cdots,s_N)}}{\mathrm{Z}^{\omega}_{\beta,h,N}}\mathrm{P}(s_1,\cdots,s_N),$$

$$\mathrm{H}^{\omega}_{\beta,h,N}(s_1,\cdots,s_n) = \sum_{n=1}^{N}(\beta\omega_n - \Lambda(\beta) + h)\,\mathbf{1}_{s_n=0} \leftarrow \text{Hamiltonian.}$$

$\mathrm{Z}^{\omega}_{\beta,h,N} \leftarrow$ partition function: fundamental to study the model.

Assumptions

- Markov chain (S,P): the return time

$$\mathrm{P}(S_n = 0, S_1 \neq 0,\cdots,S_{n-1}\neq 0) \overset{n\to\infty}{\sim} \frac{c_S}{n^{1+\alpha}}, \qquad \alpha \in (0,1)$$

- Disorder: i.i.d. sequence $(\omega,\mathbb{P})$

$$\mathbb{E}(\omega_1) = 0, \quad \mathbb{V}(\omega_1) = 1, \quad \Lambda(\beta) := \log\mathbb{E}(e^{\beta\omega_1}) < \infty.$$

Problem: study the typical trajectories of S under $\mathrm{P}^{\omega}_{\beta,h,N}$ when N is large
$\rightarrow$ localization/de-localization around the distinguished state

Critical curve

Consider the expected number of contacts

$$\mathbb{E}\mathrm{E}^{\omega}_{\beta,h,N}\left(\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{S_n=0}\right) = \frac{\partial}{\partial h}\frac{1}{N}\mathbb{E}\left[\log \mathrm{Z}^{\omega}_{\beta,h,N}\right]$$

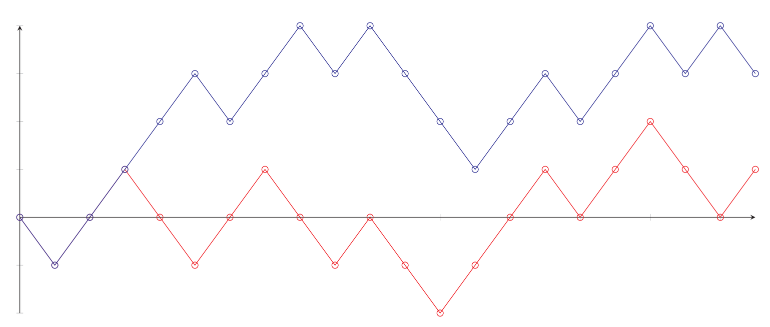
Free energy: $\mathrm{F}(\beta,h) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\log \mathrm{Z}^{\omega}_{\beta,h,N}\right] \geq 0.$

$\frac{\partial}{\partial h}\mathrm{F}(\beta,h) \rightarrow$ "limit density of number of contacts"

There exists a critical phase transition

$$h_c(\beta) = \sup\{h : F(h,\beta) = 0\}$$

It depends only on the law of the disorder.



- $h > h_c(\beta)$ localization: $\frac{\partial}{\partial h}\mathrm{F}(\beta,h) > 0$,
- $h < h_c(\beta)$ de-localization: $\frac{\partial}{\partial h}\mathrm{F}(\beta,h) = 0$.

Aim: describe $h_c(\beta)$ $\rightarrow$ when the the disorder is weak (β small), does it still influence the Markov chain?

Homogeneous model (without disorder: constant rewards/penalties) provides lower and upper bound for the critical curve:

$$h_c(0) \leq h_c(\beta) \leq h_c(0) + \Lambda(\beta),$$

and if β is small enough, then

- $h_c(\beta) = h_c(0)$, if $\alpha \in (0,1/2)$ (irrelevant disorder),
- $h_c(\beta) > h_c(0)$, if $\alpha \in (1/2,1)$ (relevant disorder).

Main results

Problem: get the critical phase transition in the weak disorder regime when $\alpha \in (1/2,1)$.

Known results: nomatching upper and lower bounds [AZ], i.e. for any $\alpha \in (1/2,1)$ there exist $\beta_0 > 0$ and $c > 0$ such that for any $\beta < \beta_0$

$$c^{-1}\beta^{\frac{2\alpha}{2\alpha-1}} \leq h_c(\beta) \leq c\beta^{\frac{2\alpha}{2\alpha-1}}.$$

Our main result provides the precise asymptotics:

$$h_c(\beta) \overset{\beta\to 0}{\sim} \mathrm{h}_c^{(\alpha)}(1)\, c_S^{-\frac{1}{2\alpha-1}}\,\beta^{\frac{2\alpha}{2\alpha-1}}.$$

The constant accounts for the value of α (through a *universal term* $\mathrm{h}_c^{(\alpha)}(1)$) as well as the specific Markov chain choice (through c_S).

Proof

Our approach uses the *continuum partition function* [CSZ]:

- Fix $\alpha \in (1/2,1)$ and rescale $\beta = \beta_N$ and $h = h_N$ with N,

$$\beta_N = \hat{\beta}\, c_S N^{\frac{1}{2}-\alpha}, \quad h_N = \hat{h}\, c_S N^{-\alpha}, \quad \hat{\beta} \in \mathbb{R}_+, \quad \hat{h} \in \mathbb{R}.$$

Then there exists a *continuum partition function* $\mathrm{Z}^{(\alpha),W}_{\hat{\beta},\hat{h},t}$ such that

$$\mathrm{Z}^{(\alpha),W}_{\hat{\beta},\hat{h},t} \overset{(d)}{=} \lim_{N\to\infty}\mathrm{Z}^{\omega}_{\beta_N,h_N,\lfloor Nt\rfloor}.$$

The continuum partition function depends on a continuum disorder W (Brownian Motion).

We prove:

- There exists the *continuum free energy*

$$\mathrm{F}^{(\alpha)}(\hat{\beta},\hat{h}) = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\log \mathrm{Z}^{(\alpha),W}_{\hat{\beta},\hat{h},t}\right],$$

- *It is comparable to the free energy*: precisely for any $\rho \in (0,1)$ there exists $\beta_0 > 0$ such that for any $\beta < \beta_0$ and $\hat{h} \in \mathbb{R}$

$$\mathrm{F}^{(\alpha)}\left(1, \hat{h}(1+\rho)^{-1}\right)\left(c_S^{-1}\beta\right)^{\frac{2}{2\alpha-1}} \leq \mathrm{F}\left(\beta, \hat{h}\, c_S^{-\frac{1}{2\alpha-1}}\beta^{\frac{2\alpha}{2\alpha-1}}\right),$$

$$\mathrm{F}^{(\alpha)}\left(1, \hat{h}(1+\rho)\right)\left(c_S^{-1}\beta\right)^{\frac{2}{2\alpha-1}} \geq \mathrm{F}\left(\beta, \hat{h}\, c_S^{-\frac{1}{2\alpha-1}}\beta^{\frac{2\alpha}{2\alpha-1}}\right).$$

- We define the continuum critical phase transition

$$\mathrm{h}_c^{(\alpha)}(\hat{\beta}) = \sup\{\hat{h} : \mathrm{F}^{(\alpha)}(\hat{\beta},\hat{h}) = 0\}$$

and the bounds on the free energy imply the asymptotics for the critical phase transition $h_c(\beta)$. □

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